Zsuzsanna Benkő (Ed.)

INTEGRATED PROGRAMMES FOR LOWER-PRIMARY TEACHER TRAINING

MATHEMATICS AND FOSTERING TALENT

Lisbon – Szeged – Vienna
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Contents

Conceptual umbrella

Since the development of hierarchical societies most social scientists agree that inequalities in the countries of the developed world are still characteristically defined by this hierarchical arrangement. Different classes, strata and groups of the society are arranged along financial, property, educational, social division of labour and settlement differences. Social scientists agree also on the fact that opportunities in the society are largely influenced by factors that can not be represented in the hierarchical arrangement. What are these factors?

Without hierarchical ordering of the factors in question here we think of inequalities related to gender: women in all societies of the developed world are in worse social positions than men of similar stratification characteristics. The national and ethnic palette of European societies is very colourful; belonging to a national or ethnic minority can be source of social disadvantages. Though cases are different by countries and societies, the state of the disabled further expands the system of unequal opportunities.

An increasingly indispensable part of knowledge-based society, highlighted more and more in the recent years – beside disciplinary knowledge (geography, biology, chemistry, etc.) – is the need for a complex orientation in the society. Public education can support social orientation the best if it is able to develop, strengthen and implement practice-oriented skills and expertise. What are those skills we aimed at developing in this project? We intend to deepen health promotion, health-conscious behaviour, value system and practice. Opportunities for health are described also along social hierarchy and along the level of education as a decisive factor within, and this chance or opportunity is further increased or decreased by minority status or of being different. It is already an important aim and task in itself that lower-primary school educators and through these educators, children, parents and grandparents should become acquainted with modern approach to health. So as to reduce health inequalities new educational and psychological methodological elements are required beside essential knowledge, because paradigm-change is also essential for getting to know, accepting and make others accept "difference".

We can hope for a modest result only if we base health promotion solely on knowledge transmission – however thorough it is –, and it is the same case if we focus only on knowledge transmission in relation to minority issues. In the case of health promotion we considered practice-oriented education and the transmission of values as decisive educational methods. In case of minority issues value is carried and transmitted by arts. Music, dance and literature – saga as a genre corresponding to the age characteristics – make this complex world visible, audible, perceptible and enjoyable. This scope of questions means not only the historically determined, traditional minority groups (for example the Gypsies, the Jews, Black people or the disabled) but it also means the present migration process within the European Union that happens in front of our eyes: migration from close countries within Europe and from distant countries in Asia, Africa, Latin-America.

There are considerable differences among European countries in terms of their attitudes to their down-and-out, and here we mean the disabled primarily. Though we know it well that those countries where disability is visible in the whole society, because most of them left the closed institutes two or three decades ago (Great Britain, the United States), has defined many tasks for themselves, this process has just started in other countries, where at the same time all this should be made acceptable for small children and for their parents, that is for the whole society.

Inequality appears not only in health chances or in case of minorities but in fostering talent as well. Hence we are well aware that public education strives at the capital-reproduction of the middle class of the society (see the theory of Bourdieu). That is why it is essential to foster the talent of those children in lower-primary school whose parents lack this cultural capital: their communication is underdeveloped, their self-evaluation is low. That is why we have chosen an educational domain for fostering talent where these two factors are not decisive, and it is mathematics. But we are not approaching mathematics the way as the vast majority of the adult society would imagine on the basis of their earlier school experiences. All these of course do not help the communication difficulties of disadvantaged children. Communication skills practice is the answer to that. According to our experiences, teacher training in most European countries does not offer this kind of skills development practice for their students.

We take an invaluable step by developing mother tongue communication, but this increases equal opportunities within the given nation only. Speaking foreign languages (like English and German), especially in three small EU countries like Hungary, Portugal and Austria, is extremely important. Austria has helped a lot in working out the German language module.

Striving at equal opportunities is of key-value in each content module. Creativity, practice-orientation, the socially integrated individual (keeping family, settlement and cultural backgrounds in mind during the education of lower-primary pupils) are the methodological bases of the transmission of these contents and values. This new perspective can form an organic part of the traditional values of lower-primary teacher training of these three countries.

Zsuzsanna Benkő
International professional coordinator of the project

Preface

The level of mathematics education is high in our countries (Hungary, Austria, Portugal), with an overcrowded curriculum in many cases. Because of the set amount of materials and the low number of lessons lower primary teachers have only a little time left for fostering talent and for differentiated development. This is extremely important for the socially disadvantaged, talented children, as their opportunities can be promoted by proper treatment. Later forms of fostering talent like contests or tasks to be sent to different journals are at the disposal of lower primary children to a lesser extent. The task of the lower primary teacher would be to raise and keep the interest of 1st and 2nd graders for mathematics, and then from grade 3 to direct this interest at contests and task-solving, in those cases especially where the family does not do it. This way we could arrive at the result that not only those children access the different forms of extracurricular talent fostering who are already interested and determined concerning mathematics but those children as well who do not feel obliged to it because of their disadvantaged position.

It is an important point that not necessarily the child who fulfils mathematical tasks quickly and without mistakes at school is talented in mathematics. Lower primary teachers should be prepared for those abilities (organising, combinative) and features in thinking (divergent, convergent, inverse) that are the signals of talent. Accordingly we should ensure the experience of success on some fields for all children who are interested (for example there are children whose space perception is more developed, and there are ones who are more skilled in combinatorial or logic tasks). It is easy to mix speed with talent, especially in case of lower primary children where reading and writing can also cause problems; several examples prove that a slow person can also be very talented.

According to our opinion lower primary teachers can have the possibility to foster talent care at school. Group work of children and prepared task sheets would help a lot to reach this aim. We would like to prepare lower primary teacher trainees for these methods and for the development of such kind of task sheets. Our aim was not to teach some new exercises or theorems, but develop the problem solving ability, critical and creative thinking of future teachers, and as a result their pupils', by doing some activities. We think that it is very important to be able to find mathematical problems in the environment of pupils which are interesting and motivating for them. Task sheets are organised not according to the mathematical content only. Problems are attached to everyday life and topics of interest for the pupils. One task sheet processes one topic.

We summarize our point of view in “Guiding Principles”, and speak about teacher’s work with pupils in primary school in “Mathematics and Fostering Talent”. We show the mathematical background of some conjuring tricks (1.), and how we can create mathematical problems from the adventures of Harry Potter (10.). The activities about ancient counting and number systems (2. 3. 4.) help to deepen the understanding of the concept of number and operations, and the rules of divisibility. The activities: Calendar (5.), Gold on a scale (6.), Chess board (11.) and Sports (12.) are in close relation in their themes to real life or tales and games, at the same time with combinatorics and number theory. The activities on geometry (7. 8. 9) give some possibilities for manipulative activities, some of them may be done with computer.

We hope that after this course future teachers will be able to construct similar activities for their pupils in primary school.

Klára Pintér
Sub-coordinator

Guiding Principles and Training Methodology

Lurdes Serrazina– José Tomás Gomes (Portugal)

Underlying our teacher education is the idea that, while learning mathematics on a teacher education course is important, it is even more important to develop an interest in research and an inquiring attitude towards the subject. The future teacher should develop a receptive attitude towards experimentation and innovation. It follows that priority should be given not to the *quantity* of mathematics itself but to the *quality* of the activities in which the future teachers are involved. As SELTER (1997) affirms, teachers become real professionals as they teach and reflect upon their teaching. Thus, the main aim of teacher education should be that the future teachers prepare themselves and involve themselves in their own professional development, so that they continue to develop throughout their career.

It is obvious that the teacher must know mathematics; but he also needs to feel comfortable in what he is teaching. To this end he has to know the mathematical concepts, techniques and procedures that apply to this level of schooling. He needs to be clearly aware of the great Mathematical ideas and of their relevance in today's world. He needs to be conversant with all the developments in the mathematics curriculum for the first cycle of basic education, as well as with those mathematical ideas that lend themselves to pre-school education (PONTE & SERRAZINA 2000). Or, in the words of BALL (1991), the future teacher needs a sound understanding of mathematics that goes beyond tacit knowledge of the "know-how" type and becomes an explicit knowledge. This involves being able to *discuss* mathematics, i.e. not just knowing the steps required to follow an algorithm, but being able to explain the rationale applied and the meanings and reasons for certain relationships and procedures. For this author, explicit knowledge of mathematics implies not merely saying the names of mathematical propositions and formulae, but should include language that goes beyond the superficial representation. Explicit knowledge involves reasons and relationships: being able to explain *why,* as well as being able to relate particular ideas or procedures with others inside mathematics.

In exactly the same way as mathematics in basic education is learned by doing, so the professional skills of the future mathematics teacher are acquired through the carrying out of a whole gamut of activities. These should be present on three levels: on the level of mathematics in basic education; on the level of teaching activities involving mathematics in basic education, and on the level of theoretical activity in the domain of mathematics education theory. On each of these levels, only if there is time for reflection will activity lead to skills development. Throughout the course, a line of development should be followed, ranging from practical activities to reflection on these activities, observation of others' practical activities and analysis of their reflections, and analysis of the theory behind one's own activities and those of others (SERRAZINA 2002).

It follows that teacher training should not consist of teaching recipes and methods that can simply be applied directly in the classroom, but should, first and foremost, help future teachers to develop their autonomy. This involves helping them to increase their mathematical knowledge, to learn how to teach mathematics (how children learn mathematics), to assess the quality of teaching materials, etc.

It must be borne in mind that students, on arrival at teacher-training colleges, have many years' experience as pupils of mathematics themselves, and have formed their own ideas of mathematics and its teaching. These beliefs and points of view are often unconscious, and therefore not accessible either to the students themselves or to their teachers.

The process of reflection is essential since, as research has shown, it is not unusual for future teachers to leave teacher-training colleges with their early attitudes to the subject unaltered (BULLOGH 1997). They pass through teacher education college dealing with their dilemmas as best they can, but retaining their beliefs and their conceptions of what it is to be a teacher and longing for the day when they will have their own class. They frequently work within the constraints imposed upon them while realizing that this is not the way they will work in their own classroom.

Future teachers should therefore be provided, throughout their training, with mathematical experiences that will develop, through a historical and cultural approach, perspectives on the nature of mathematics designed to enhance their predisposition for the subject and the self-confidence to study mathematics on their own; they should be provided with opportunities for problem solving and for research in mathematics. Special attention should be paid to the role of the new technologies and their application in the chosen activities.

The main option should be based on the idea that there must be consistence between the teacher education model and the didactic model. As asserted by GARCIA (1995), the teacher educator should "teach to teach by teaching". That is, there must be coherence between the education model used by the teacher educator in his classes and the didactic model he wishes to transmit to the future teachers. The teacher educator who wants his trainees consciously to develop a didactic model should be careful to incorporate in his own practices those principles he wishes to instil in his students.

As already mentioned, teachers tend to teach in the same way as they were taught (LESTER et al. 1994). We might say that future teachers and educators have an implicit model – a *content* knowledge, acquired during their own schooling, of the mathematics they will have to teach – and a *didactic* model, gleaned from their own experience as pupils. This knowledge is usually rigid, partial, fraught with misapprehensions and, in most cases almost exclusively applied to the skills of calculus, with no regard for their application to problems in daily life. EVEN & LAPPAN (1994) consider that future teachers in general see mathematics as a set of fixed rules totally unrelated to each other, their teaching as a process of communication, and their learning as memorization.

According to the *Professional Norms* published by the NCTM (1994), "future teachers' past experience when they were at school has a profound effect on the education they give their pupils" (p.130). Breaking with the inertia constructed during the years of schooling, and changing conceptions, implies instilling a different attitude towards knowing, doing, learning and teaching mathematics. "Those with whom they are learning are models that can help to bring about a change in attitude towards what mathematics is and how it is learned" (p.130). Trainees learn from those responsible for their education, who in their turn, consciously or unconsciously, embrace a particular conception of teaching and learning, that is, the didactic reference model underlying their activity as educator and which is implicitly transmitted through their action. In this context, authors like VACC & BRIGHT (1994) cite one of the most significant results of their work with future teachers: the importance for the successful outcome of the training process of coherence and consistence between the trainer's *modus operandi* (underlying didactic model) and the philosophy he tries to transmit through the syllabus content (explicit didactic model).

The challenge confronting us as trainers is compounded by the fact that these concepts are essentially tacit in nature and that we therefore need, as a *sine qua non* to bring about change, instruments which will help to make them explicit. Moreover, the fact that it is an implicit knowledge, tightly bound up with the students' personal experiences, means that it is deeply ingrained and hard to change. Awareness of the preconceptions of future teachers, throughout the training process, therefore constitutes a further basic principle of training. Initial training must not only attempt to make explicit the tacit knowledge of future teachers, but also ensure that this knowledge evolves through reflective processes centred on the tackling and solving of problems, that is, research. There is a need to create an environment of constant inquiry in which provocative discussions can arise and actively involve all students. Learning environments in which future teachers are given the opportunity to use physical and material models, calculators and computers, enhance the essential bases for the construction of wide-ranging, in-depth knowledge of mathematical concepts and procedures. In such a learning environment, future teachers will be given experiences that will form the foundation on which they will build learning environments for their own pupils.

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Mathematics and Fostering Talent

Margarete Grimus (Austria)

1. Mathematics in Primary School

The supportive culture in compulsory schools

The Concept of Fostering Talent in Primary School

Supportive culture is the guarantee that every child will receive professional guidance in all areas of his/her life and education. This support covers social, intellectual, motor, emotional, creative and perceptual skills.

Individualized teaching requires more emphasis on "open education" and teaching through projects (learning using all senses). Likewise, it calls for teaching, which focuses on creativity, allowing for questions, experiments and mistakes while catering to different interests.

Key principles: Pupils should increasingly assume more personal responsibility for their own learning progress. In this regard, the development of strategies suited to specific needs and learning abilities is an essential prerequisite for the acquisition of knowledge and skills. Priority should be placed on a variety of pupil-centered teaching and learning methods.

Pedagogical and organizational results: education based on a wider range of learning fields as well as flexibility in the choice of teaching and learning methods with priority on active and pupil-centred learning methods.

2. Mathematic-didactic Discussions

Social neutrality and universality are frequently attributed to mathematics. Thus, the perception of mathematics as a cultural product, and consequently as a socially integrated product, is often lost (GELLERT 1999).

Three key criteria are considered to be relevant for teaching (according to GELLERT):

1. In mathematics classes it is essential to impart a basic law of mathematical concepts and methods. In primary school, numbers are primarily ascribed "a pragmatic value," although they are actually meant to provide "practical help for everyday life" based on "authentic, real-life situations". (HEYMANN 1996 quoted by GELLERT).

2. The development of technical skills should contribute towards critical reasoning which enables societal problems to be analyzed using mathematics. To this end, suitable special methods should be used or reality should be represented propadeutically through mathematical models. This promotes the development of special thought patterns such as "interconnected, self-responsible and anticipatory thinking skills" which are adapted to the "complexity of our society" and to the "increasingly rapid technical and economical changes".
3. Mathematics lessons should be conducive to the development of a creative and responsible personality. This relates to the inclusion of ludic and artistic elements as special teaching methods for mathematics.

Mathematics is more than only arithmetic skills.

- The requirements for basic skills have evolved due to pocket calculators and computers, the perception of arithmetic as a fundamental skill has undergone a radical change. Nevertheless, every child, adolescent and adult needs a sensible, rudimentary knowledge of mathematics for daily life and work.
- Learning mathematics can be sustainably promoted through experimental activities.
- For critics, the danger of applying the concept of "active, discovery-based learning" and "productive drills" lies in the resulting disassociation in primary school classes of fundamental aspects of mathematics such as mathematizing and reasoning.
- Money is viewed as a quantity just like weights and measures, since the price of goods as a measurable quantity does not justify any distinction between economic assessments and physical measurements.

3. Teaching Mathematics

For the purposes of practical classroom instruction, the idea that teachers have of mathematics is considered to be decisive. In fact, teachers' beliefs and conceptions about a subject are more important for the provision of practical classroom instruction than the subject field itself. (THOMPSON, 1992). Various empirical studies[1] have shown, "that an algorithmic understanding of mathematics prevails among primary school teachers and consequently, the goal of teaching is for the pupils to master basic calculation operations such as the multiplication tables

[1] In this regard, GELLERT quotes from BOBIS and CUSWORTH 1995; GELLERT 1998; SCHUCK 1995; and FREUDENTHAL 1991.

from 1–10 and written subtraction. Such lessons often lead to a perception of mathematics as a schematical-mechanical exercise. Math instruction may therefore be characterized as a class where mathematical operations i.e. algorithms are strung to each other without any focus on relevance or aim. In this way, math instruction in primary school itself becomes an algorithm. In this meta-algorithm, *algorithm follows algorithm,* with liturgical precision as in an unalterable ritual" (GELLERT 1998).

For this reason, GELLERT suggests math teaching experiences be reflected upon for application in primary school teacher training and that as a result thereof, the narrow approach to math instruction with emphasis on drilling arithmetic operations be replaced by an intellectual decision-making process.

The mystique that surrounds mathematical drills - along with authoritative teachers and textbooks – has led mathematical truths to be viewed as immutable, extrinsically determined, universal and pure (cf. GELLERT, no year).

The mystique that surrounds mathematical solutions relates to the fact that a solution can invariably be determined in an unequivocal manner and that judgments as to the correctness of attempted solutions can be equally unequivocal. This conflicts with the view that logical, reasoned argumentation, including asking questions, is desirable and even essential.

The outcome of these reflections is the "Math 2000"[2] concept, which has been specifically developed for primary school. Math 2000[3] contains many exercises and different problem situations for active-discovery learning. The exercises go a step further than merely solving individual exercises, by enabling the discovery of laws or patterns, which in turn allow for problem-solving strategies with the broadest application possible.

3.1. Mathematics should be experienced as an activity.

Likewise, pre-service teachers should also experience in teacher training that mathematics is not "out of touch with reality", and is therefore not merely a "theory" to be discarded later; at the same time that the tendency to underestimate primary school mathematics should be counteracted. Specialized skills are essential when it comes to making competent decisions at several different educational levels: discovering, reasoning, mathematizing and representing are also general learning targets for teacher training.

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[2] C. WITTMANN and GERHARD MÜLLER initiated this project at the University of Dortmund in cooperation with schools.

[3] http://www.mathematik.uni-dortmund.de/didaktik/mathe2000/

It is also of utmost importance that children be given the opportunity to experience mathematics as an activity, which draws on intuitiveness, imagination and creative thinking. Moreover, math should be viewed as a subject enabling them to gain insights and understanding by stimulating independent and collective thought, and where they also make their own discoveries which in turn reinforces confidence in their own thinking capacity and intellectual pleasure." (SPIEGEL 1995:199).

Therefore teaching training should include the following points:

- Becoming familiar with handling basic mathematical concepts and the didactics of teaching mathematics at the elementary level.
- Mathematics lessons at the elementary and the lower secondary school level should not be looked down upon as an "unreasonable burden".
- They should rather be regarded as an essential foundation for extensive math instruction in primary schools; it would be useful for pre-service teachers to "revisit" their mathematics education and to also refresh their knowledge in this area as need be. (This could also be done through self-study).
- The additional experience gained, should enable a greater capacity for combining knowledge input from mathematics, psychology and pedagogy with the ultimate aim of being able to formulate and carry out a personal, well-founded position with regard to learning and teaching mathematics in primary school. (cf. GELLERT 1995).

Teacher training should provide teacher trainees with the opportunity of making mathematical discoveries and of realizing their relevance in order to render it possible for their pupils to make similar experiences later on.

One issue to be brought up for discussion is teachers' expectations for certain answers: this limits the way teachers perceive pupils' comments and consequently affects the answers given by pupils. Independent thinking has little place if pupils are to orientate themselves in their own reflections on the basis of messages and statements from teachers who dominate discussions (LOSKA 2001).

4. Mathematical Literacy

Here emphasis is not placed on calculating or applying formulas, but in fact on the ability to:

- acknowledge and understand the role that mathematics play in the world
- make well-founded mathematical judgments
- approach mathematics in a way that corresponds to people's present and future needs in their lives as productive, dedicated and reflecting citizens (cf. PISA 2000:18).

5. Mathematics in the Context of the New Learning Culture

Information society, lifelong learning and new learning culture

Current discussions on educational policy increasingly reflect the need to develop specific inter-disciplinary competence in addition to specialized skills. In addition, research in the area of teaching/learning focuses nowadays on self-paced learning, general problem-solving skills and social competence. These key skills have an inter-disciplinarian nature, because they are not tied to specific situations or subjects. They should be adaptable to inter-disciplinarian situations and applicable to new problems. In fact, along these lines, topics may be integrated into mathematics lessons for developing skills and capabilities based on a mathematical mode of thinking.

Conventional textbooks are often structured around "concise" clearly defined illustrative texts or brief transformation exercises, which have nothing to do with real life. The findings of the TIMMS study[4] as well as the PISA study[5] show that certain teaching methods are problematic from the point of view of subsequent use of knowledge: Exercises without any context are "speedily" solved and factual knowledge is often rated as being more important than understanding. The focus is on learning performance rather than on the learning process. (cf. HENSE/MANDL/GRÄSL).

The special features of the new culture are that children and adolescents are given the opportunity to apply their subjective interests[6] independently in solving real problems in various economic areas. During the problem-solving process, they are then able to view mathematics as a useful "language", which enables them to model their approaches to problem-solving This modelling phase should be carried out as group work.

- Interests can be understood to be a network of connections between objects and persons. These subjective person-object relations are aimed at gaining insight, and associated with positive feelings.
- Interests are learned over and above a genetic predisposition (especially during early childhood) and then remain relatively stable.
- "Interests can broaden during learning processes. Learning processes which relate to the subjective interests of pupils – which are therefore (int-

[4] Third International Mathematics and Science Study, Dritte Internationale Mathematik- und Naturwissenschaftsstudie – TIMSS http://www.timss.mpg.de/

[5] Programme for International Student Assessment PISA Project center Austria http://www.sbg.ac.at/assess/pisa/pisa-home.htm (5.7.2003)

[6] Real Problems and subjective interests http://www.schule.suedtirol.it/blikk/angebote/modellmathe/ma1804.htm

rinsically) motivated by the person-object relation, and thus also linked to a previously existing subjective knowledge construction, are more successful than learning processes which are rendered "attractive" from the outside (extrinsically motivated)" (BIKNER-AHSBAHS 1998:125).

- Frequently, at the beginning of a learning sequence, facts are "couched in" for learning motivation, but thereafter no longer play a role in subsequent lessons. Pupils can quickly see through this and it soon loses its meaning. "On the basis of empirical studies, Deci and Ryan (IPN Kiel) have found out that intrinsic motivation can be easily developed, if the activity in the lesson is relevant, if the lesson contains options and if the class atmosphere allows for feelings to be expressed." (BIKNER-AHSBAHS 1998:125).

This means that when teaching mathematics:

- Interests are to be initiated and promoted
- Interest orientation must be seen as an aspect of teaching on the long-term
- Individualization and diversity are to be sought: the diversity of the mathematical phenomenon is to be fully tapped in order to give as many pupils as possible the opportunity to learn mathematics with motivation (ibid).

Three phases: modeling, systematization and application.

Problem-solving work concerning various real problems should be carried out in small groups and moderated by teachers in such a way that all problems can be dealt with by children and adolescents based on much the same mathematical model or "language pattern". In the systematization phase (which follows the modeling phase), it is possible, by specific ordering to engineer the discovery that all small groups have used much the same mathematics for their solution. Mathematics "originates" (appears in one's mind) as an abstraction of various realities.

Thus it can be incorporated into previously developed (learned) mathematical structures. "Mathematics thus gradually becomes a self-styled scientific system".

The systematization phase is followed by an application phase, which is also a practical phase. This represents the individualized phase of learning whereby basic mathematical knowledge is consolidated, so it can be used over and over again as frequently as possible in various "applications". What are known as "couched exercises" are thus relevant in this context. A basic knowledge of mathematics is thus acquired which is not only important for subsequent math classes, but which may be of lasting importance for life.

The results of research into teaching/learning provide good arguments in favour of a radical change in our educational culture. In this area, the use of computers and the Internet are also to be considered for learning purposes. Multimedia offers opportunities for dynamic and constructive ways to prepare and present one's own material. In addition, much factual information based on figures can be very clearly illustrated by diagrams.

Learning and working with digital media should also be seen in light of benefits for social interaction. Priority should be placed on working in pairs. However, to ensure effective cooperation and constructive learning for all, the exercise must be structured in a way that really makes working together indispensable although individual performance must still be visible and verifiable.

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Activities

Compiled and worked out by Klára Pintér–Tamás Árki–Lurdes Serrazina–José Tomás Gomes

1. Conjuring tricks (Klára Pintér)

The conjuring tricks are to be played in a way, that the conjuror presents the trick many times, in various ways perhaps. If someone from the audience thinks he knows the solution of the trick then it's his turn to perform it, but the solution should not be told to the others. This way more and more pupils can find the clue to the trick.

1. The audience thinks of an animal shown in the picture. The conjuror goes along the lines, anti-clockwise. The audience silently spells the names of the animals, thinking of a letter by every step of the conjuror (a double letter or a diphthong counts as two, and the start point is the first step). When the audience thinks of the last letter, it gives a sign of readiness, and the conjuror is pointing at the given animal the audience was thinking of.

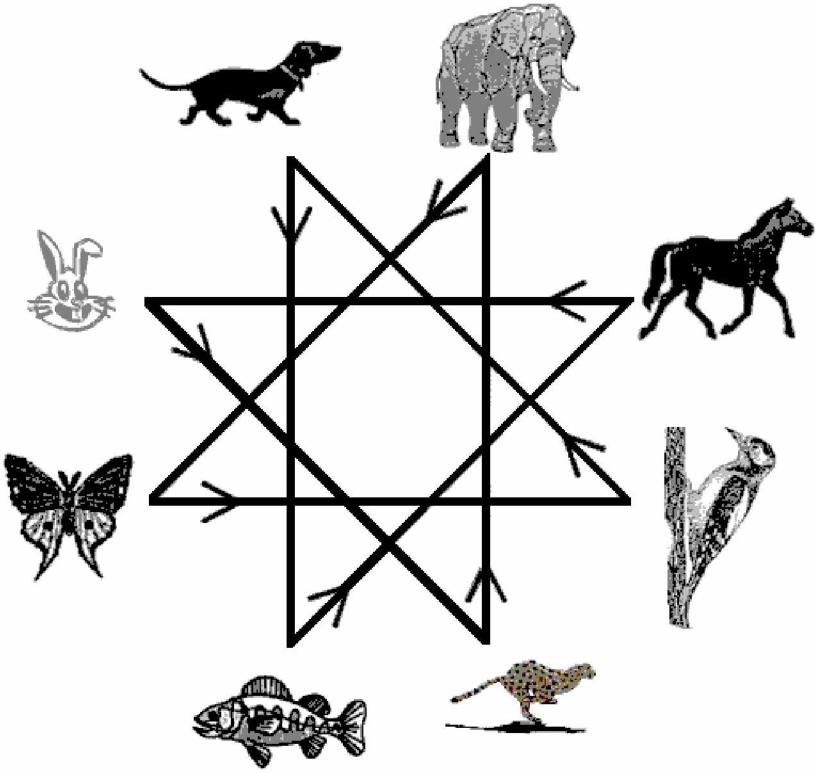
   The figure shows the English version of the conjuring trick.

   Dog, fish, horse, rabbit, leopard, elephant, butterfly, woodpecker.

   Let's solve the trick and make a similar one to other languages or to plants, objects, etc. as well.

   Solution

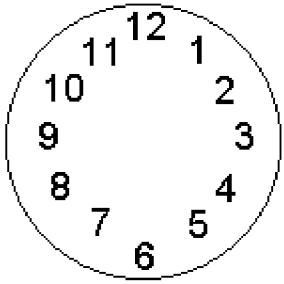
   We should start from the butterfly (this is the first step with the first letter), then with the third step we reach to the dog, then the number of letters in the name of each animal increases by one. We should follow the arrows. As the number of letters consisting the name of the animals is 4, 5, 6, 7, 8, 9, respectively, the number of moves until we reach an animal is as much as many letters there are in its name.



(In the Hungarian version the animals may be in order: horse, fish, mouse, dog, cat, leopard, butterfly, water-snake)

2. The audience thinks of a number on the face of the clock. The conjuror points at a number with a knock, when the audience adds one to the number it was thinking of. The same procedure is repeated for the next knock, and so on. When the audience reaches at 20, the conjuror is pointing exactly at the number the audience was thinking of.

   This can be played in a way, that the members of audience agree on the number they are thinking of, they give a signal when they are at 19, the conjuror starts to concentrate hard, and for the 20th points at the required number. It can also be played in a way that everybody thinks of a different number on the clock, and the conjuror "guesses" everybody's number, i.e. when someone reaches 20 in counting, the conjuror is pointing at his or her number.



Solution

When the audience reaches at 19, it doesn't matter which numbers the conjuror was pointing at so far, what matters is that the conjuror counted silently the number of knocks. If there were n knocks before 19 was said, then the required number is $19-n$, this should be pointed at the next knock. The conjuror has an easier job if he starts from 19, and counts backwards with each knock. This way when the audience signals, he is exactly at the required number.

If the conjuror tries to guess everybody's number at the same time, the following track of thought can be followed. The largest number the audience could think of is 12, so by 8 knocks later is the earliest time the conjuror has to guess the number of someone, namely the number of the person who thought of 12. So it doesn't matter where the first 7 knocks pointed at, what matters is that the 8^{th} knock should point at 12, the next one to 11, and so on, one-by-one, anti-clockwise. This way the conjuror will "guess" everybody's number.

3. Everybody places 9 cards on the table, in a 3x3 square form. Starting from the card in the middle always make as much moves to a border card as much the conjuror says. You can move to and fro, up and down, as you like. Then the conjuror tells which card should be removed. Interestingly, no one is standing on that card at that moment. Then the conjuror tells again how many moves should be made. The rules say that one can not step on the place of a missing card, and one can not jump over an empty place. Go on like this, until only one card remains.

   One possibility for the conjuror to conduct the game:

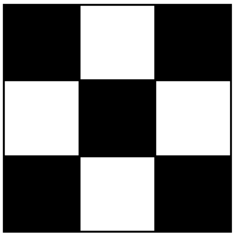
   Make 7 moves and remove A1. (Rows are marked by A, B, C, columns are marked by 1, 2, 3.)

   Make 8 moves and remove A3.

   Make 9 moves and remove C2

Make 5 moves and remove C1.
Make 6 moves and remove C3.
Make 7 moves and remove B1.
Make 4 moves and remove A2.
Make 6 moves and remove B3.
Interestingly, finally everybody is standing on the middle card, where he or she started from.
Instead of using cards, everybody can draw a 3x3 square and move on that. It is advisable that the conjuror also displays the removed cards, for example with the help of post-it stickers put on the blackboard. It is more spectacular if we create a computer program for it.
This trick was performed some years ago on TV by David Copperfield in a way that everybody had to move on cards shown on the television display, and as a result of "concentrating", finally everybody was standing on the same card.

Solution
Colour the little squares (cards) of the 3x3 square black and white (chess-board like), the neighbouring squares are of different colour (the middle square should be black).

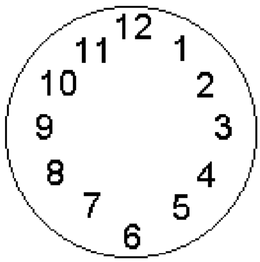


We move to a contrasting colour by each step, so after an even number of moves we are standing on a square of the same colour we started from, and after an odd number of moves we are standing on a contrasting colour. So regardless of the moves one takes, everybody is standing on the same colour. The conjuror always knows the colour of the square the audience is standing on, and he can remove any of the contrasting squares, but should pay attention to the requirement, that the remaining squares should be connected.
On the basis of this he can conduct the game as he likes, providing possibilities to guess the trick. Of course it is not a requirement that the last square should be the starting point exactly.

There are two very useful ideas in the key of this conjuring trick:

- the squares are coloured in a chess-board like fashion;
- the colour of the square we are stepping on is different according to the parity of the number of moves.

4. The audience thinks of a number on the face of a clock, then skips forward clockwise one by one starting from 12 as much letters as there are in the given number. In the next move the audience skips as much as many letters consist the number it reached at in the previous step, and so on. In the meantime the conjuror can remove numbers where the audience can not skip to. Of course, he can not remove numbers where someone is standing at. The numbers will finally disappear and everybody will stand on the same number. Let's work the trick out for more languages!



Solution

We consider the moves in case of all the possible numbers one can think of.
These moves are more self-evident than the ones in the previous trick.
One version in English:

Number in question	*In letters*	*After the 1st move*	*After the 2nd move*	*After the 3rd move*	*After the 4th move*	*After the 5th move*
1	One	3	8	1	6	3
2	Two	3	8	1	6	3
3	Three	5	9	1	6	3
4	Four	4	8	1	6	3
5	Five	4	8	1	6	3
6	Six	3	8	1	6	3
7	Seven	5	9	1	6	3
8	Eight	5	9	1	6	3
9	Nine	4	8	1	6	3
10	Ten	3	8	1	6	3
11	Eleven	6	9	1	6	3
12	twelve	6	9	1	6	3
Removed numbers				2, 4, 8, 12	1, 5, 7, 10	6, 9, 11

Finally everybody is standing on number 3. The English version is quite simple, as after 3 moves everybody will stand on the same number, without removing any numbers. In case of other languages this can be obtained only by removing some numbers.

One version of the Hungarian trick – the double letters or diphthongs are counted as two letters:

Number in question	*In letters*	*After the 1st move*	*After the 2nd move*	*After the 3rd move*	*After the 4th move*	*After the 5th move*	*After the 6th move*
1	Egy	3	8	1	4	9	4
2	Kettő	5	7	10	3	9	4
3	Három	5	7	10	3	9	4
4	Négy	4	8	1	4	9	4
5	Öt	2	7	10	3	9	4
6	Hat	3	8	1	4	9	4
7	Hét	3	8	1	4	9	4
8	Nyolc	5	7	10	3	9	4
9	Kilenc	6	9	3	9	9	4
10	Tíz	3	8	1	4	9	4
11	Tizenegy	8	1	4	9	9	4
12	Tizenkettő	10	1	4	9	9	4
Removed numbers				7, 11, 12	1, 2, 10	5, 8	3, 6, 9

A possible German version:

Number in question	*In letters*	*After the 1st move*	*After the 2nd move*	*After the 3rd move*	*After the 4th move*	*After the 5th move*
1	Ein	3	7	2	2	2
2	Zwei	4	8	1	2	2
3	Drei	4	8	1	2	2
4	Vier	4	8	1	2	2
5	Fünf	4	8	1	2	2
6	Sechs	5	9	2	2	2
7	Sieben	6	11	2	2	2
8	Acht	4	8	1	2	2
9	Neun	4	8	1	2	2
10	Zehn	4	8	1	2	2
11	Elf	3	7	2	2	2
12	Zwölf	5	8	1	2	2
Removed numbers			4, 5, 6, 10	3, 7, 8, 9, 11, 12	1	

A possible French version:

Number in question	*In letters*	*After the 1st move*	*After the 2nd move*	*After the 3rd move*	*After the 4th move*	*After the 5th move*	*After the 6th move*
1	Un	2	6	12	9	9	9
2	Deux	4	12	6	12	3	9
3	Trois	5	10	3	12	3	9
4	Quatre	6	10	3	12	3	9
5	Cinq	4	12	6	12	3	9
6	Six	3	9	3	12	3	9
7	Sept	4	12	6	12	3	9
8	Huit	4	12	6	12	3	9
9	Neuf	4	12	6	12	3	9
10	Dix	3	9	3	12	3	9
11	Onze	4	12	6	12	3	9
12	Douze	5	10	3	12	3	9
Removed numbers		1, 8, 11	7	4	2, 5, 10	6, 12	3

A possible Italian version:

Number in question	*In letters*	*After the 1st move*	*After the 2nd move*	*After the 3rd move*	*After the 4th move*	*After the 5th move*	*After the 6th move*	*After the 7th move*
1	Uno	3	6	11	11	1	10	1
2	Due	3	6	11	11	1	10	1
3	Tre	3	6	11	11	1	10	1
4	Quattro	7	12	6	1	10	10	1
5	Cinque	6	9	1	6	1	10	1
6	Sei	3	6	11	11	1	10	1
7	Sette	5	11	5	5	6	1	1
8	Otto	4	11	5	5	6	1	1
9	Nove	4	11	5	5	6	1	1
10	Dieci	5	11	5	5	6	1	1
11	Undici	6	9	1	6	1	10	1
12	Dodici	6	9	1	6	1	10	1
Removed numbers			7, 8	2, 3, 9, 12	4		5, 6	10, 11

These tricks can be further developed, for example we can think of making the trick 5 or 6 moves long in each language.

One of the most important elements of conjuring tricks is motivation. During guessing the trick behind a conjuring presentation the ability to observe, systematise and realise regularity develops.

Tricks 1 and 4 can also help in language learning through the spelling of numbers, animal names, etc. These tricks can not be translated word by word to other languages, as the course of the tricks is the function of the number of letters consisting a word. So the trick has to be reconsidered for an other language and this improves the ability to construct as well.

Reference

GARDNER, M. (1956): Mathematics magic and mystery. Dover

2. Ancient counting (Klára Pintér)

Counting has accompanied human culture throughout history. In the beginning pebbles, knots tied on ropes (e.g. China) denoted the numbers. At that time there were as many signs as the number of countable items. For example, when counting the animals people cut as many lines on a stick as many animals they had.

Denoting larger numbers was too complicated, so in a next counting system different symbols were applied to different number groups. One of the most ancient counting system types was born in the Egyptian culture.

2.1. The Egyptian system

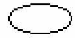
2.1.1. Egyptian numbers

In the system based on one-shot grouping, different symbols were applied to different number groups, and the position and order of symbols did not affect the number it represented.

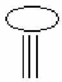
The Egyptian system is based on grouping by tens, so this is a 10 based number system; different symbols were applied for the different powers of 10.

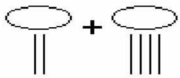
Egyptian sign	*Name*	*Value*	*10 exponent*
	Stroke	1	10^0
	heel bone	10	10^1
	coiled rope	100	10^2
	Lotus flower	1000	10^3
	pointed finger	10000	10^4
	pollywog	100000	10^5
	astonished man	1000000	10^6

The symbol and the name of number groups was of course not uniform, this is an easily drawable version. For example 12 was written like this:  , or  , or  .

Egyptians knew fractions; they depicted each fraction as a sum of fractions the numerator of which is 1. To denote these, they wrote the number corresponding to the denominator under the sign  .

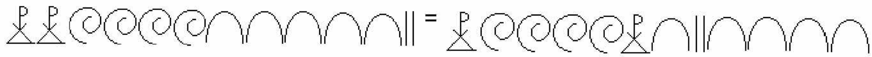
For example:

$\frac{1}{3}=$  ;

$\frac{3}{4}=\frac{1}{2}+\frac{1}{4}=$  .

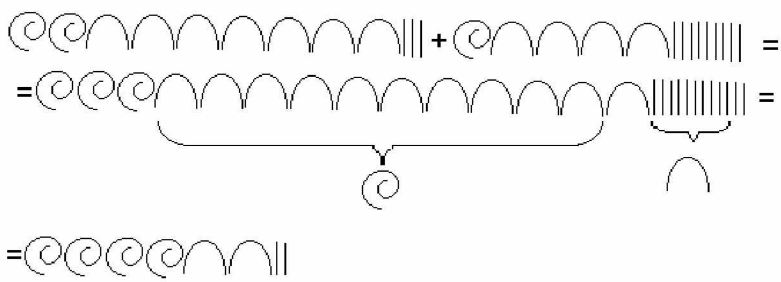
When reading a number, the consisting symbols should simply be added, regardless to their order.

For example:

2452= 

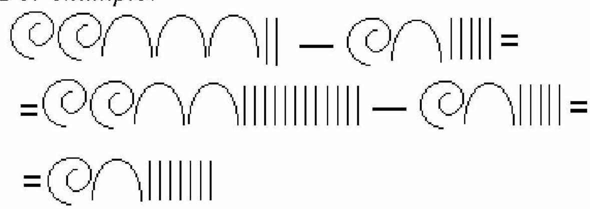
In making additions we consider the consisting symbols together, then make the possible exchanges.

For example:



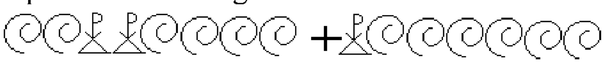
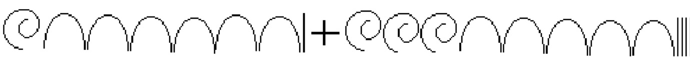
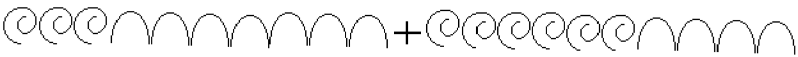
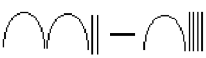
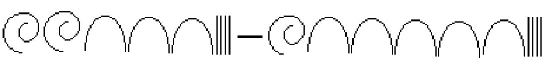
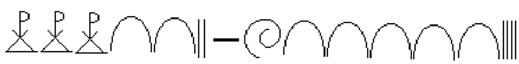
In case of extractions it can happen, that there are less in the minuend than in the subtractive; in this case a  is exchanged to 10 pieces of , and the subtraction can be completed.

For example:



The same methods are applied in case of place value method of writing numbers as well, though when we execute written operations, the usual, mechanical technique hides the method that would be important for lower-primary teachers.

Exercises

1. Write the following numbers with the help of Egyptian symbols!
   a) 25 b) 253 c) 14532 d) 2003 e) 100001
2. Which numbers are denoted by the following Egyptian symbols?
   a) 
   b) 
   c) 
   d) 
3. Complete the following additions!
   a) 
   b) 
   c) 
4. Complete the following subtractions!
   a) 
   b) 
   c) 

2.1.2. Egyptian multiplication

The technique of Egyptian multiplication is based on the fact that it is easy to duplicate symbols in Egyptian writing, as well. Similar methods were applied in case of other nations. The method is demonstrated on two numbers, with the use of Arabic digits to make it easier.

Let us complete the following multiplication: 14 x 23.

We write the numbers in two columns, the first starts with 1, the second with 23, and the succeeding number is obtained by doubling the one above. We stop when among the powers of 2 in the first column we find some that add up to 14. These powers of 2 are marked with an * , and by adding up the numbers in the other column of the same row we obtain the product. The procedure is shown in the table below.

1		23	
* 2	→	46	46
* 4	→	92	92
* 8	→	184	+184
			322

8+4+2=14

The basis of the method is the conversion of the multiplier into number system 2, then the reorganisation of the multiplication based on distributivity. It is very important methodologically, as lower primary teachers have to realise how the place value method of writing appears in written multiplication, and that multiplication is distributive to addition.

Exercises

Complete the following multiplications by the Egyptian way!

1. 11 x 38;
2. 24 x 51;
3. 53 x 94.

2.1.3. Egyptian fractions

Egyptians used fractions with numerator 1, so called stem-fractions, and were striving to express other fractions as sum of stem-fractions with different denominator. They created tables for writing the $\frac{2}{n}$ type fractions.

Problem

Write the following fractions as the sum of stem-fractions with different denominators:

$$\frac{2}{3};\ \frac{2}{5};\ \frac{2}{7};\ \frac{2}{11};\ \frac{2}{13};\ \frac{2}{35}.$$

Solution

Decomposition can often be done in various ways. Try to find a method generally applicable during decomposition! We denote numerator 1 with a line above the number, similar to the Egyptian sign.

$$\frac{2}{3}=\overline{2}+\overline{6};$$
$$\frac{2}{5}=\overline{3}+\overline{15};$$
$$\frac{2}{7}=\overline{4}+\overline{28};$$
$$\frac{2}{11}=\overline{6}+\overline{66};$$
$$\frac{2}{13}=\overline{8}+\overline{52}+\overline{104};$$
$$\frac{2}{35}=\overline{30}+\overline{42}=\overline{18}+\overline{630}=\overline{21}+\overline{105}=\overline{20}+\overline{140}.$$

The decomposition of $\frac{2}{3}$ through division by k provides a good method for an arbitrary denominator divisible by 3: $\frac{2}{3k}=\frac{1}{2k}+\frac{1}{6k}$. For an odd denominator the $\frac{2}{2k-1}=\frac{1}{k}+\frac{1}{k(2k-1)}$ decomposition always works, but of course this is not the only possible decomposition. We don't have to consider even denominators, as we get a fraction with numerator 1 after reduction to a common denominator.

It can be proved that there are extremely many ways of decomposing a fraction into a sum of stem-fractions with different denominators. Only one method is presented here to show how can be obtained more and more, and finally all denominators become different when there are still common denominators in the decomposition of a fraction into stem-fractions:

$$\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)}.$$

Trials of decomposition into stem-fractions provide good experiences to counting with fractions and realising general rules.

Exercise

Decompose the following fractions into the sum of stem-fractions!

$$\frac{3}{42};\quad \frac{5}{11};\quad \frac{3}{17};\quad \frac{7}{25}.$$

2. 2. Multiplicative counting system

In the number system used by the Egyptians the symbols sometimes had to be written down many times, for example 99 was coded by 18 signs. Multiplicative systems help to overcome this problem. Here we apply grouping and different symbols for each group on the one hand, and these symbols can be multiplied through a multiplier written in front, on the other. Such a multiplication system was developed at the Greeks. They used letters to denote number groups:

I (ióta)	1
Γ (gamma)	5
Δ (delta)	10
H (eta)	100
X (khí)	1000
M (mű)	10000

Here 5 is denoted by separate symbol, so for example $9=\Gamma IIII$. $50=5\cdot 10$ was denoted by a $\Gamma=5$ written in front of $\Delta=10$: $50=\overline{|\Delta}$. Similarly, $500=\overline{|H}$; $5000=\overline{|X}$; $50000=\overline{|M}$.

For example:

$1989=X\overline{|H}HHHH\overline{|\Delta}\Delta\Delta\Delta\Gamma IIII$.

Exercises

1. How would you write the following Greek numbers nowadays?
   a) $\overline{|M}MXXX\overline{|H}HH\overline{|\Delta}\Delta\Gamma IIII$
   b) $MM\overline{|X}HHH\overline{|\Delta}\Delta\Delta\Delta\Gamma II$
2. Use Greek numbers to denote the following numbers!
   a) 78; b) 193; c) 2587; d) 53892.

2.3. Place value method of number presentation

In case of place value number writing we use different symbols to denote the multiples of the number groups. The position of the symbols shows the multiple of the number group we are working in. The presently used Hindu-Arabic number writing, based on number system 10, operates in this system. An earlier way of number writing based on the same principles was the Babylonian number system. The grouping was done by the 60's, and from right to left there were symbol groups showing the number of decreasing exponents of 60 in the number. Two signs were applied: 1=▼, and 10=◀, to denote all the numbers smaller than 60.

For example:

6=▼▼▼▼▼▼*;* *12=*◀▼▼*;* *35=*◀◀◀▼▼▼▼▼.

There are two place value in numbers between 60 and $3600=60^2$; the first shows how many times 60 is in the number, the second shows how many 1 is in the number, and so on.

Example:

▼▼ ◀◀◀▼▼▼

$2 \bullet 60 + 33 \bullet 1 = 153.$

▼▼ ◀◀▼ ◀▼▼▼▼▼

$2 \cdot 60^2 + 21 \cdot 60 + 15 = 7200 + 1260 + 15 = 8475.$

Zero was not marked in Babylonian number system at the beginning, that caused misunderstandings in the place value system, as open spaces between the digits did not always marked definitely the presence of zero at one of the place values.

Exercises

1. Which numbers are denoted by the following Babylonian symbols?
   a) ▼◀◀▼▼▼▼▼▼▼▼;
   b) ◀◀◀◀◀▼▼▼◀◀▼▼▼▼▼▼▼;
   c) ▼▼▼ ◀◀▼▼▼▼▼◀.

2. Write the following numbers in the Babylonian system!

   a) 963; *b) 4895;* *c) 9742;* *d) 25894.*

3. Write numbers that could be depicted by the following symbols, observing that it is not self-evident that the open spaces between the place values denote a zero or not!

   ▼◀◀▼▼◀◀◀▼▼▼▼▼

References

CAJARI, F. (1993): A history of mathematical notations. Dover

GILLINGS, R. J. (1972): Mathematics in time of the pharaohs. Dover

3. Number systems (Klára Pintér)

Front, front, front, front,
Front, front, front, middle,
Front, front, front, end,

Front, front, middle, front,
Front, front, middle, middle,
Front, front, middle, end,

Front, front, end, front,
Front, front, end, middle,
Front, front, end, end,

Front, middle, front, front,
Front, middle, front, middle,
Front, middle, front, end,

Front, middle, middle, front,
Front, middle, middle, middle,
Front, middle, middle, end,

Front, middle, end, front,
Front, middle, end, middle,
Front, middle, end, end,

Front, end, front, front,
Front, end, front, middle,
Front, end, front, end,

Front, end, middle, front,
Front, end, middle, middle,
Front, end, middle, end,

Front, end, end, front,
Front, end, end, middle,
Front, end, end, end,

Middle, front, front, front,
Middle, front, front, middle,
Middle, front, front, end,

Middle, front, middle, front,
Middle, front, middle, middle,
Middle, front, middle, end,

…..

3.1. Counting

3.1.1. Problem

The pirates have sacked a lot of gold coins on their latest journey, and they want to count them. When they started to count the gold one-by-one, they always made a mistake, that's why they were looking for a more precise method.

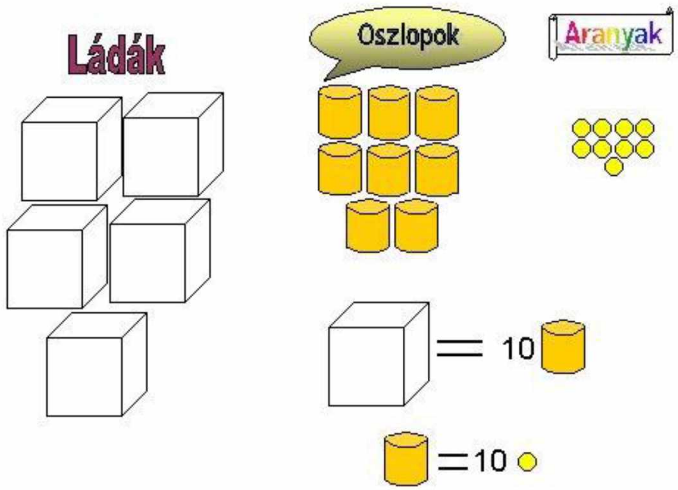
I. Solution

Here is a possible way of counting, supposing the pirates had 589 golden coins.

The pirates are quite good at counting up to 10, that is why they put all the coins to columns of 10 at first. 9 coins remained.

The number of 10-coin columns was still too much, so they put the columns into cases by tens. 8 columns remained.

Now they can count the cases; there are 5 cases.



(Ládák = cases; Oszlopok = columns; Aranyak = golden coins)

The number of golden coins is:

5 cases plus 8 columns plus 9 coins

$= \mathbf{5} \cdot 10 \cdot 10 + \mathbf{8} \cdot 10 + \mathbf{9} = 589$ coins, in number system based on 10.

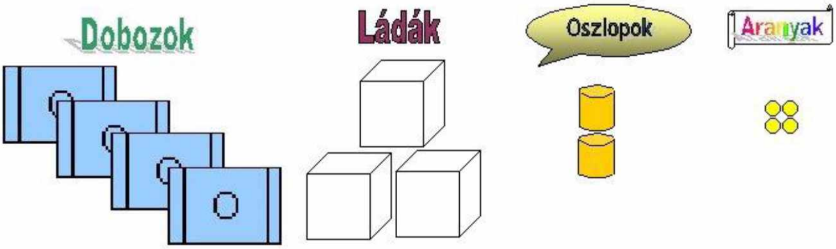
II. Solution

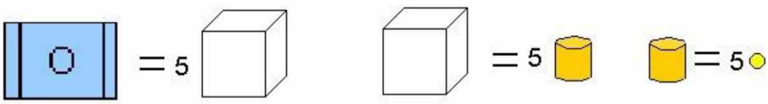
If the pirates can not safely count up to 10 either, then let's distribute the coins into columns of 5 first. 4 coins remain.

Let's put the columns of 5 into cases, 5 columns to each case. 2 columns remain.

Let’s put 5 cases into one box. 3 cases remain.

Finally we got 4 boxes.





(Dobozok = boxes, Ládák = cases; Oszlopok = columns; Aranyak = golden coins)

4 boxes + 3 cases + 2 columns + 4 coins

$\mathbf{4} \cdot 5 \cdot 5 \cdot 5 + \mathbf{3} \cdot 5 \cdot 5 + \mathbf{2} \cdot 5 + \mathbf{4}$

This number is 4324_5. in number system based on 5.

Computing it in number system based on 10: $4{\cdot}125+3{\cdot}25+2{\cdot}5+4 = 589$.

III. Solution

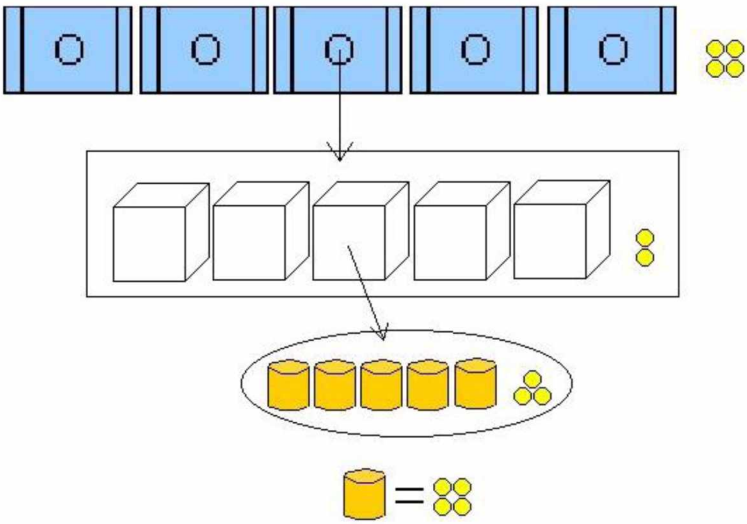
The pirates find 5 boxes and 5 cases.

Let’s distribute the coins evenly into the boxes, for example one by one to each box. There are 4 coins left, and there are an equal number of coins in each box, though this number is still too large to be counted one by one.

Let’s distribute the coins of one box into the 5 cases evenly. 2 coins remain, and there are an equal number of coins in each case.

Let’s distribute the coins of a case into 5 columns evenly. 3 coins are left, and there are an equal number of coins in each column.

There are 4 coins in each column.



1 column	4 golden coins
1 case	5 columns plus 3 coins = $5 \cdot 4 + 3 = 23$ coins
1 box	5 cases plus 2 coins = $5 \cdot (5 \cdot 4 + 3) + 2 = 117$ coins
Altogether	5 boxes plus 4 coins =
	$= 5 \cdot [5 \cdot (5 \cdot 4+3)+2]+4 = 5 \cdot 5 \cdot 5 \cdot \mathbf{4} + 5 \cdot 5 \cdot \mathbf{3} + 5 \cdot \mathbf{2} + \mathbf{4}$!
	$= 5 \cdot 117 + 4 = 589$ coins in number system based on 10.
	The same number in number system based on 5 would be 4324_5.

The digits of number system based on 5 are: 0; 1; 2; 3; 4
5 is the base number of the number system, we put it into the lower right corner of the number: e.g. 4324_5

The place value table of number system based on 5, if we give the place values in number system 10:

$5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ = 3 125	$5 \cdot 5 \cdot 5 \cdot 5$ = 625	$5 \cdot 5 \cdot 5$ =125	$5 \cdot 5$ =25	5	1

3.1.2. Problem

Convert the number 42031_5 into number system based on 10.

Solution

Based on the place value table:

$42031_5 = 4 \cdot 625 + 2 \cdot 125 + 0 \cdot 25 + 3 \cdot 5 + 1 = 2500 + 250 + 0 + 15 + 1 = 2766_{10}$

3.1.3. Problem

Convert the number 1341_{10} into number system based on 5.

I. Solution

Based on the place value table:

The largest place value less than the above number is: 625.

$1341_{10} = 2 \cdot 625 + 91$	divide the number by 625, the quotient is 2, the remainder is 91
$91 = 0 \cdot 125 + 91$	divide the remainder by 125, the quotient is 0, the remainder is 91
$91 = 3 \cdot 25 + 16$	divide the remainder by 25, the quotient is 3, the remainder is 16
$16 = 3 \cdot 5 + 4$	divide the remainder by 5, the quotient is 3, the remainder is 4
$4 = 4 \cdot 1 + 0$	

By this way $1341_{10} = 2 \cdot 625 + 0 \cdot 125 + 3 \cdot 25 + 3 \cdot 5 + 4 \cdot 1 = 20331_5$

II. Solution

Like the pirates counted the coins in the III. Solution of the problem.

$1341 = 5 \cdot 268 + 1$	divide the number by 5, the quotient is 268, the remainder is 1
$268 = 5 \cdot 53 + 3$	divide the quotient by 5, the quotient is 53, the remainder is 3
$53 = 5 \cdot 10 + 3$	divide the quotient by 5, the quotient is 10, the remainder is 3
$10 = 5 \cdot 2 + 0$	divide the quotient by 5, the quotient is 2, the remainder is 0
$2 = 5 \cdot 0 + 2$	divide the quotient by 0, so we are ready, the remainder is 2

$$1341_{10} = 5 \cdot (5 \cdot (5 \cdot (5 \cdot 2 + 0) + 3) + 3) + 1 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 2 + 5 \cdot 5 \cdot 5 \cdot 0 + 5 \cdot 5 \cdot 3 + 5 \cdot 3 + 1 =$$
$$= 20331_5$$

Exercises

1. Visiting the giant's castle Jack discovered that the giant applies a unique counting method. As he was counting his golden eggs, Jack heard the followings:
   Tir, tel, top, tum, tak, tirtak, teltak, toptak, tumtak, taktak, tirtaktak, teltaktak, toptaktak, …
   What does the giant say at the 20^{th}, 27^{th} and 32^{nd} egg?
2. Convert the following numbers number system based on 5 into number system based on 10:
   12; 34; 42; 102; 403; 342; 1042; 2310; 1423; 3222; 10423; 24130;

3. Convert the following numbers in number system based on 10 into number system based on 5:
   12, 50; 34; 86; 74; 131; 263; 587; 756; 1497
4. Give the date of your birthday in number system based on 5!
5. Give today's date in number system based on 5!
6. You receive an e-mail from an other planet:
   I would like to meet a child who is 22 years old, similar to me. I attend grade 11 at school. I have 3 brothers and sisters; they are 24, 13 and 4 years old. My mom is 124 years old. My father is 2 years older than she, which means he is 131 years old. I am waiting for your mail. Trixi
   How old do you think the family members of Trixi are?
   What is the age difference among the children?
   Answer Trixi in his number system!

3.2. Number system 12

Earlier number system based on 12 was also in use, the remnants of which are:

1 dozen = 12 pieces 1 year = 12 months

3.2.1. Problem

Steve is 12 years and 3 months old, his brother Andrew is 2 years and 10 months older than him. Steve's friend, Pete, is 11 months younger than Steve. How old (years and months) are Andrew and Pete?

Solution

12 years 3 months + 2 years 10 months = 14 years 13 months = 15 years 1 months
So Andrew is 15 years and 1 month old.
12 years 3 months – 11 months = 11 years 15 months – 11 months = 11 years 3 months.
So Pete is 11 years and 3 months old.

3.2.2. Problem

Let 1 dozendozen be equal 12 dozen and 1 dozendozendozen be equal 12 dozendozen.
The school has bought 2 dozendozendozen sketching-papers, 3 dozendozen paint-brushes, 8 pieces of paint, and 5 dozen pencils for the drawing club. How many items have they bought altogether?

Solution

The number of paints: 8

The number of pencils: 5 dozen = $5 \cdot 12 = 60$
The number of paint-brushes: 3 dozendozen = $3 \cdot 12$ dozen = $3 \cdot 12 \cdot 12 = 432$
The number of sketching-papers: 2 dozendozendozen = $2 \cdot 12$ dozendozen = $2 \cdot 12 \cdot 12$ dozen =$2 \cdot 12 \cdot 12 \cdot 12$=3456
Altogether 3456+432+60+8 = 3956 items were bought.

Digits of number system based on 12 are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B
Letters are used to denote "digits" corresponding to 10 and 11: $10_{10}= A_{12}$ and $11_{10}= B_{12}$

3.2.3. Problem
Let us list the positive divisors of 10_{12}!

Solution
10 is divisible by 1 and 10.

$\boxed{1 \quad 2}$ $\boxed{3 \quad 4}$ $\boxed{5 \quad 6}$ $\boxed{7 \quad 8}$ $\boxed{9 \quad A}$ $\boxed{B \quad 10}$

10 is divisible by 2

$\boxed{1 \quad 2 \quad 3}$ $\boxed{4 \quad 5 \quad 6}$ $\boxed{7 \quad 8 \quad 9}$ $\boxed{A \quad B \quad 10}$

10 is divisible by 3

$\boxed{1 \quad 2 \quad 3 \quad 4}$ $\boxed{5 \quad 6 \quad 7 \quad 8}$ $\boxed{9 \quad A \quad B \quad 10}$

10 is divisible by 4

$\boxed{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}$ $\boxed{7 \quad 8 \quad 9 \quad A \quad B \quad 10}$

10 is divisible by 6

So the positive divisors of 10_{12} are: 1, 2, 3, 4, 6, 10_{12}
Divisibility rules in number system based on 12_{10} which are according to the last digit.

3.2.4. Problem
Let us count in number system based on 12_{10} by $10_{12,}$ starting from 0.

Solution
0; 10; 20; 30; 40; 50; 60; 70; 80; 90, A0; B0; 100; 110; 120; 130; 140; 150, 160; 170; 180; 190; 1A0; 1B0; 200; …

We have listed numbers divisible by $10_{12} = 12_{10}$.

In number system based on 12_{10} exactly those numbers are divisible by $10_{12} = 12_{10}$, that end to 0.

Each number that is divisible by $10_{12} = 12_{10}$ is divisible by 2, 3, 4, and 6 also.

3.2.5. Problem

Let's decide if $45A86_{12}$ is divisible by 2, 3, 4, and 6.

Solution

$$\underbrace{45A86 = 45A80}_{\text{divisible by } 10_{12}} + \underbrace{6}_{\substack{\text{divisible by 2, 3, 6} \\ \text{not divisible by 4}}}$$

So, $45A86_{12}$ is divisible by 2, 3, and 6, but it is not divisible by 4.

A number in number system based on 12 is divisible by 2, 3, 4, and 6 only when the digit standing at the place of the ones is divisible by 2, 3, 4, and 6.

Exactly those numbers are divisible by 2 whose end is: 0 or 2 or 4 or 6 or 8 or A.
Exactly those numbers are divisible by 3 whose end is: 0 or 3 or 6.
Exactly those numbers are divisible by 4 whose end is: 0 or 4 or 8.
Exactly those numbers are divisible by 6 whose end is: 0 or 6.

Exercises

1. How many months old is the child who is 7 years and 5 months old?
2. How many months ago were you born?
3. How many years old is the child who has celebrated his 139 months old birthday this year?
4. Peter has counted the number of cherries he has gathered from the tree. He has counted 5 dozendozendozen, 11 dozendozen, 10 dozen and 7 pieces of cherry. How many pieces of cherry did Peter gather?
5. The 1 litre soft drink boxes are put to cardboard boxes by the 12. 12 cardboard boxes are put into a chest. How many chests, how many cardboard boxes and how many boxes of soft drinks should someone buy, if 800 litres of soft drinks are needed at a sport competition?
6. Convert the following numbers in number system based on 12 into number system based on 10:
   391; 57A; 28B; AB;
7. Convert the following number numbers in system based on 10 into number system based on 12:
   89; 142; 156; 567;

8. In number system based on 12 which whole number follows 9B, and which stands in front of 100 if we count one by one?
9. Which numbers are divisible by 2, 3, 4, 6, or 12?
   8_{12}; $31A_{12}$; 563_{12}; $6B8_{12}$; $A0_{12}$; $4B_{12}$; 2222_{12};
10. How many ways are there to complete the number 3A*4* to be
    (a) divisible by 3 and 2.
    (b) divisible by 3 and 4.
    (c) divisible by 6 and 4.
    (d) divisible by 4, but not divisible by 2.
    (e) divisible by 6, but not divisible by 3.
    (f) divisible by 2 or 3.
    (g) divisible by 3 or 4.
11. How many four-digit numbers in number system based on 12 can be made out of digits 3, 4, 6, B, if each digit can be used only once? How many of these are divisible by 4? How many of these are divisible by 3? How many of these are divisible by 12?
12. Which number system did we formulate 76_{10} in?
    a) 2211 (b) 1001100 (c) 301 (d) 204 (e) 1030 (f) 84
13. In which number system was the following letter written?
    „I was born 110 years ago. I went to school at the age of 21, and I am at the same school for 12 years already. I have 220 class-mates, and we are counting in number system 10, which might be a little bit strange for you. This is good because we have to know only 10 digits, and our addition and multiplication table is easier than yours. My favourite subject is Mathematics, but unfortunately we have only 11 lessons a week, that's why I attend afternoon classes too. I have entered the intergalactic Mathematics competition, but I have fluffed 12 exercises out of 1010, so I got only the 11th place."
    Translate the letter into our number system!
14. Replace the different letters with different digits in a way to make the following addition true in number system based on 3!

(a)
$$\begin{array}{r} ACA \\ +\ AAB \\ \hline BBC \end{array}$$

(b)
$$\begin{array}{r} ABA \\ +\ ABA \\ \hline ABAB \end{array}$$

Look for any other number systems in which the above problem can be solved.

15. In which number system are the following equations true?
    (a) 6+1 = 10 (b) 44 + 1 = 100 (c) 100 + 1 = 101
    (d) 210+10=220 (e) 11+11 =110 (f) 10+10+10+10 = 40
    (g) 1+1+1=10 (h) 2+2+2+2 = 22 (i) 3+3+3+3 = 13

(j) $100+100=1000$ (k) $1000+1000=2000$ (l) $1+2+3+4 = 20$
(m) $10+10=100$ (n) $10 \cdot 10 = 100$ (o) $1000 : 10 = 100$

16. The same number was written down in two different number systems. Which are these two number systems, if
(a) $1000_a=20_b$ (b) $1101_c = 31_d$ (c) $1000_e=30_f$ (d) $1101_g=41_h$
(e) $10\,000_x= 100_y$

17. Put the following numbers (formulated in different number systems) into an ascending order!
110110_2; 2112_3, 211_5; 321_4; 1101_4

18. Put the following numbers into a decreasing order!
a= the smallest 6-digit number in number system based on 2
b= the smallest 3-digit number in number system based on 4
c= the smallest 3-digit number in number system based on 5
d= the smallest 4-digit number in number system based on 4.

19. Put the following numbers into ascending order!
A= the largest 6-digit number in number system based on 2
B= the largest 3-digit number in number system based on 4
C= the largest 3-digit number in number system based on 5
D= the largest 4-digit number in number system based on 4.

20. Consider the number system based on 3, 4 or 10. Find natural numbers the last digit of which is 2, such that by moving the last digit to the first position, we obtain the double of this number.

21. (a) How many 4-digit numbers are there in number system based on 3?
(b) How many 4-digit numbers can be created in number system based on 3 by using number cards containing the numbers 2; 1; 1; and 0?
(c) How many 4-digit numbers can be created in number system based on 3 using the digits 0 or 1? How many numbers in number system based on 2 are defined by these?

Interesting facts: In 2000 BC a Chinese product was found, containing 64 different symbols. Each symbol contained 6 horizontal, parallel lines, some of which had a hole in the middle, some of which didn't. The symbols were decoded by LEIBNIZ (1646–1716), a German mathematician. Each of the symbols is a number in number system based on 2, that is 6 digits long and zeros can be at the beginning also. A line with a hole in the middle denoted the digit 0, the other line denoted 1. The digits of the number are to be read from the bottom to the top.

4. **Planet Five** (Klára Pintér)

Number system base on 5 is used on Planet Five. We do not know any other digits in this lesson than 0; 1; 2; 3; and 4, and we can count only in number system based on 5. As we often, if involuntarily, shift back to number system 10, in same cases – we found most critical – we made a remark that the given number is to be considered in number system 5.

4.1. Numbers on Planet Five

The numbers are one by one:

0; 1; 2; 3; 4; 10; 11; 12; 13; 14; 20; 21; 22; 23; 24; 30; 31; 32; 33; 34; 40; 41; 42; 43; 44, 100; 101; 102; 103; 104; 110, …

The *place value table* on Planet Five:

10000_5	1000_5	100_5	10_5	1

4.2.Operations on Planet Five

Operations and their characteristics are the same on Planet Five as the ones used in our number systems.

4.2.1. Addition table

Use the number-line when creating the addition table!

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

4.2.2. Addition of numbers with more digits

(a) $\begin{array}{r} 2314 \\ +\,1130 \\ \hline 3444 \end{array}$ Addition was done by columns, starting from place value 1 towards the bigger place value. In a column the sum was not bigger than 4.

(b) $\begin{array}{r} 2431 \\ +3424 \\ \hline \end{array}$ The sum of numbers at place value 1 is: 1+4=10, write 0 down and take1 on.

11410 The sum at place value 10_5 is: 3+2+1=11,

write 1 down and take 1 on.

The sum at place value 100_5 is 4+4+1=14,

write 4 down and take 1 on.

The sum at place value 1000_5 is 2+3+1=11.

As there are no more columns left, write 11 down.

4.2.3. Subtraction of numbers with more digits

(a)

$$\begin{array}{r} 4324 \\ -\ 1012 \\ \hline 3312 \end{array}$$

The subtraction is done by columns, starting from place value 1, towards the bigger place value. In a column the respective digit of the minuend was bigger than the respective digit of the subtractive.

(b)

$$\begin{array}{r} 3201 \\ -\ 1342 \\ \hline 1304 \end{array}$$

The digit of the minuend is bigger than the respective digit of the subtractive, so to obtain 11, we have to add 4 to 2, write 4 down and take 1 on.

At place value 10_5, 0 should be added to 4+1=10, to get 10, Write 0 down and take 1 on.

At place value 100_5, 3 should be added to 3+1 = 4, to get 12, write 3 down and take 1 on.

At place value 1000_5, 1 should be added to 1+1 = 2, to get 3, write 1 down and we are ready.

4.2.4. Multiplication table

Let's use the addition table for creating the multiplication table:

For example: $2 \cdot 3 = 3 \cdot 2 = 3+3 = 11$,

$2 \cdot 4 = 4 \cdot 2 = 4+4 = 13$,

$3 \cdot 3 = (3+3)+3 = 11+3 = 14$,

$3 \cdot 4 = 4 \cdot 3 = (4+4)+4 = 13+4 = 22$,

$4 \cdot 4 = (4+4+4)+4 = 22+4 = 31$

·	0	1	2	3	4	10
0	0	0	0	0	0	0
1	0	1	2	3	4	10
2	0	2	4	11	13	20
3	0	3	11	14	22	30
4	0	4	13	22	31	40
10	0	10	20	30	40	100

If we multiply a natural number by 10_5, each of its digits goes to a one bigger place value, and there will be 0 at the place of ones.

4.2.5. Multiplication of a more-digits number by a one-digit number

(c) $\underline{1202} \cdot 2$
2404

(d) $\underline{2343} \cdot 3$
13134

Multiplication is done starting from place value 1 towards the bigger place value. The product of the digits was not bigger than 4 in any of the cases. Starting at place value 1: $3 \cdot 3 = 14$, write 4 down and take 1 on.
At place value 10_5: $3 \cdot 4 + 1 = 22+1 = 23$, write 3 down and take 2 on.
At place value 100_5: $3 \cdot 3 + 2 = 14+2 = 21$, write 1 down and take 2 on.
At place value 1000_5: $3 \cdot 2 + 2 = 11+2 = 13$, as there are no more digits in the multiplicand, write 13 down.

4.2.6. Multiplication of a more-digits number by a two-digit number

$2314 \cdot 34 = 2314 \cdot 30 + 2314 \cdot 4 = 2314 \cdot 3 \cdot 10 + 2314 \cdot 4$:

$$\begin{array}{r} \underline{2314} \cdot 34 \\ 130020 \\ +\underline{\ \ 20321} \\ 200341 \end{array}$$

4.2.7. Division of a more-digit number by a one-digit number.

Use the multiplication table!

We start the division at the dividend's biggest place value.

3421 : 2 = 1433	3 divided by 2 is 2, remained 1.
14	14 divided by 2 is 4, $2 \cdot 4 = 13$, so 1 remained.
12	12 divided by is 3, $2 \cdot 3 = 11$, so 1 remained
11	11 divided by 2 is 3, $2 \cdot 3 = 11$, so the remnant is 0.
0	

Exercises

1. Count on Planet Five starting from 1 first, then from 2, and finally from 3
   (a) One by one;
   (b) By two;
   (c) By three;
   (d) By five, i.e. by 10.
2. Which is the biggest number on Planet Five which has
   (a) two digits;
   (b) three digits;

(c) four digits.

3. How many positive integers are there on Planet Five that have
   (a) two digits;
   (b) three digits;
   (c) four digits.
4. How many two-digit numbers are there on Planet Five for which the sum of the digits is equal to 4?
5. How many three-digit numbers are there on Planet Five for which the sum of the digits is equal to 4?
6. Put the following numbers in number system based on 5 into ascending order:
   (a) 233; 1244; 421; 3214; 3120; 1204.
   (b) 4102; 12231; 3122; 12344; 2212; 3002.
   (c) 42342; 30201; 1022; 4432, 10322; 31000.
7. Complete the following additions:
   (a) 1303+2140 (b) 3212+131 (c) 4231+1342 (d) 2314+3423
8. Complete the following subtractions:
   (a) 4324-1213 (b) 2431-421 (c) 4231-1342 (d) 3102-2413
9. Complete the following multiplications:
   (a) $2143 \cdot 2$ (b) $3424 \cdot 4$ (c) $4312 \cdot 23$ (d) $3124 \cdot 14$
10. Find natural numbers on Planet Five the last digit of which is 2, such that by moving the last digit to the first position, we obtain the double of this number.
11. Which is the biggest six-digit number on Planet Five for which the sum of the digits is equal to 14_5?
12. Which is the smallest natural number on Planet Five for which the sum of the digits is equal to 12_5?
13. We would like to number the pages of a Planet Five book, one by one, starting with 1. How many pages can we number, if we have
    (a) 24_5
    (b) 131_5
    (c) 324_5 digits altogether?

4.3. Divisibility on Planet Five

Let's write the numbers of Planet Five into a table, one-by-one:

0	1	0	1	2	0	1	2	3
2	3	3	4	10	4	10	11	12
4	10	11	12	13	13	14	20	21
11	12	14	20	21	22	23	24	30
13	14	22	23	24	31	32	33	34
20	21	30	31	32	40	41	42	43
22	23	33	34	40	44	100	101	102
24	30	41	42	43	103	104	110	111
31	32	44	100	101	112	113	114	120
33	34	102	103	104	121	122	123	124
40	41	110	111	112	130	131	132	133
42	43	113	114	120	134	140	141	142
44	100	121	122	123	143	144	200	201
Even – odd		divided by 3			divided by 4			
		0 remainder	-1 remainder	-2 remainder	0 rem.	-1 rem.	–2 rem.	–3 rem.

Let's count from 0 by 10_5!

0; 10; 20; 30; 40; 100; 110; 120; 130; 140; 200; 210; 220; 230; 240; 300; 310; 320; 330, 340; 400; 410; 420; 430; 440; 1000; 1010; …

If we count from 0 by 10_5, we get the numbers divisible by 10_5, these all end in 0.

A number in number system based on 5 is divisible by $10_5 = 5_{10}$, if and only if it's last digit is 0.

Write the first $10_5 = 5_{10}$ row of the above table down in a way that the sum of the digits consisting the numbers is put in place of each number.

0	1	0	1	2	0	1	2	3
2	3	3	4	1	4	1	2	3
4	1	2	3	4	4	10	2	3
2	3	10	2	3	4	10	11	3
4	10	4	10	11	4	10	11	12
Even – odd		divided by 3			divided by 4			
		0 remainder	-1 remainder	-2 remainder	0 rem.	-1 rem.	–2 rem.	–3 rem.

If the sum of the digits of a number is even, then the sum of the digits of a number greater by 1 is odd.
If the sum of the digits of a number is odd, then the sum of the digits of a number greater by 1 is even.
0 is an even number.

A number in number system based on 5 is even, i.e. divisible by 2, if and only if the sum of its digits is even.

On the basis of the table one can suppose, that a Planet Five number divided by 4 results in the same remnant as the sum of its digits divided by 4 does.

A number in number system based on 5 is divisible by 4 if and only if the sum of its digits is divisible by 4.

$3241_5 = 3 \cdot 1000_5 + 2 \cdot 100_5 + 4 \cdot 10_5 + 1 = 3 \cdot (444_5 + 1) + 2 \cdot (44_5 + 1) + 4 \cdot (4+1) + 1 =$
$= 3 \cdot 444_5 + 2 \cdot 44_5 + 4 \cdot 4 + 3 + 2 + 4 + 1 = 3 \cdot 4 \cdot 111_5 + 2 \cdot 4 \cdot 11_5 + 4 \cdot 4 \cdot 1 + 3 + 2 + 4 +$
$1 = =4 \cdot (3 \cdot 111_5 + 2 \cdot 11_5 + 4 \cdot 1) + 3+2+4+1$
divisible by 4 the sum of the digits

Each number divisible by 4 is divisible by 2 as well.
(The rule of divisibility by 3 is not as simple, but you can try to figure it out.)

Exercises

1. Chose from the following Planet Five numbers the even ones!
   12; 11; 10, 30; 41; 100; 110, 34, 124, 322; 1011; 1204,
2. Choose from the following Planet Five numbers those divisible by 4!
   10, 23, 40, 22, 31, 123, 301, 112; 314; 2413; 14223;
3. Complete the following numbers to make the Planet Five number divisible by 2.
   1*; 2*1; 3*; 22*; 10*3;
4. Complete the following numbers to make the Planet Five number divisible by 4.
   2*; 31*; 1*2; 3*01; 41*; 2*12;
5. How many ways are there to write two digits in place of the asterisks if we know, that the Planet Five number is even?
   (a) 1** (b) 30*2*

6. Decide which of the following statements are true, and which are false:
   (a) Every number divisible by 4 is divisible by 2.
   (b) There is an even number divisible by 4.
   (c) Every number divisible by 2 is divisible by 4.
   (d) There is a number divisible by 4 that is odd.
   (e) If each of the digits of a Planet Five number is even, then the number is even too.
   (f) If the number of odd digits of a Planet Five number is even, then the number itself is even.
   (g) The number of even digits of each even Planet Five number is even.

7. Decide if the result of the following operations is even or odd!

(a) 1+1+1+1	(g) 42-2	(m) $3122 \cdot 3$
(b) 2+2+2	(h) 234-21	(n) $3 \cdot 1442$
(c) 3+3+4	(i) 3104-212	(o) $1423 \cdot 11$
(d) 10+11	(j) 12+431-23	(p) $32401 \cdot 201$
(e) 12+34	(k) $3412 \cdot 2$	(q) $41303 \cdot 410$
(f) 241+322	(l) $2 \cdot 10243$	(r) $14 \cdot 324 \cdot 4022$

8. How many Planet Five integers are there between 10_5 and 100_5 in which the sum of the consisting digits is even, and the sum of the digits of a number greater by one is also even? Examine the problem in number system 3, 4 and 10 also.

5. Calendar (Klára Pintér)

The following exercises have a common topic: the calendar, while the mathematical contents behind are various. The topic is especially motivating at the beginning of the year.

5.1. Problem

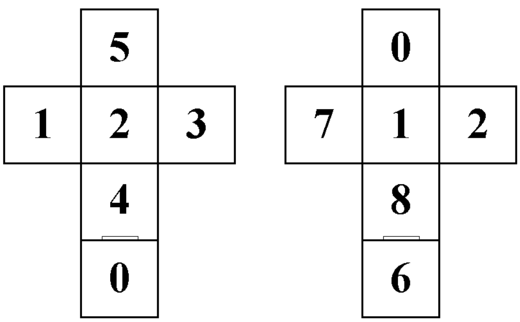
In a perpetual calendar months are presented on different sheets, and each day is marked by two cubes in a way, that each day is displayed by two digits. If a day is one digit long in the date, then the first digit is 0. How should the numbers be written to the faces of the cubes to make all these possible?

Solution

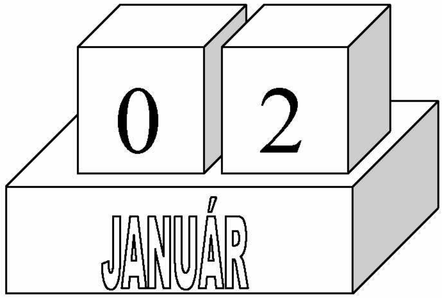
At least one piece is needed from each digit, which means 10 digits from 0 to 9. In denoting the days each digit can appear at the place value of ones after 0, 1 and 2, so these numbers should be present in both cubes. After all, if there is a 1, for example, only in one of the cubes, then if 5 would be in the same cube, we could not display 15, i.e. 1 could not be present together with digits in the same cube (and of course we could not display 11 and 22 as well). This would mean 3 more digits, so 10+3=13 digits should be displayed on the 12 faces of the two cubes, so it seems the exercise can not be solved.

But, if we realise, that number 6 written on a face can be used as a 9 if turned upside down, then we have found the way to display the digits on the faces of the two cubes in the required fashion.

For example: on the 1st cube: 0; 1; 2; 3; 4; 5 and on the 2nd cube: 0; 1; 2; 6; 7; 8.



Make the above described perpetual calendar out of a cardboard sheet or a wooden cube!



Január = January

5.2. Problem

What is the next member of the series: F28 M31 A30 M31?

Solution

The forming rule of the series is: *February is 28 days long, March* is *31* days long, *April is 30* days long, and *May is 31* days long, so the next member, considering that *June is 30* days long, is J30.

This exercise is more of a riddle than a mathematical task. It develops divergent thinking that is often needed in teaching mathematical algorithms, rules.

5.3. Problem

If the 6^{th} of January, 2003 is a Monday, which day of the week will be

(a) 18 days later?
(b) 45 days later?
(c) 2003 days later?
(d) On the 25^{th} of January?
(e) On the 14^{th} of February?
(f) On the 15^{th} of June?

Solution

(a) As a week consists of 7 days, 7 and 14 days later comes Monday again, 4 days pass then, and it is Friday.
(b) Similar to the previous one, because $45 = 6 \cdot 7 + 3$, Monday comes again in 6 weeks, i.e. $6 \cdot 7 = 42$ days, then 3 more days pass, and it will be Thursday.
(c) $2003 = 286 \cdot 7 + 1$, so 2003 days later it will be Tuesday.

(d) The 6^{th}, 13^{th}, 20^{th} and 27^{th} of January will be a Monday, by counting backwards we get, that the 25^{th} of January is Saturday.

(e) Following the previously presented method, 31^{st} of January is Friday, 1^{st} of February is Saturday, so the 15^{th} of February is Saturday too, and the 14^{th} of February is Friday.

It is faster to apply the method we used in the first 3 cases. Until the 14^{th} of February $31 - 6 + 14 = 39$ days pass from the 6^{th} of January, $39 = 5 \cdot 7 + 4$, so the 14^{th} of February will come 4 days after Monday, i.e. it will be a Friday.

(f) Until the 15^{th} of June $31 - 6 + 28 + 31 + 30 + 31 + 15 = 160$ days will pass, $160 = 22 \cdot 7 + 6$, so the 15^{th} of June is a Sunday.

These exercises serve to acquire experiences in the usefulness of remainders of dividing by seven. The days of the week represent the remainder classes of 7.

5.4. Problem

Which was the last year when there wasn't a Friday 13?

Solution

First we create a table where the dates of Fridays are presented, when the year is not a leap-year, if the 1^{st} of January is a Friday:

January	*Februar*	*March*	*April*	*May*	*June*	*July*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
1	5	5	2	7	4	2	6	3	1	5	3
8	12	12	9	14	11	9	**13**	10	8	12	10
15	19	19	16	21	18	16	20	17	15	19	17
22	26	26	23	28	25	23	27	24	22	26	24
29			30			30			29		31

It is clear from the table, that if the 1^{st} of January is a Friday, then there will be a Friday 13 in August.

If the 1^{st} of January is a Saturday, then the table shows the Saturdays. Friday 13 comes if there is a Saturday 14, i.e. in May. The same procedure should be applied if the year starts on an other day. As Friday is displayed in the original table on days 13, 14, 15, 16,17, 18, and 19, whichever day the beginning of the year is, there will always be a month where there is a Friday 13, due to the 7 kinds of shift.

A similar table can show also, that there will surely be a Friday 13 in a leap-year as well.

We could have made the table for any other year starting with any other day, and there any 7 consequent dates could have been picked to prove, there is always a Friday 13 in every year.

To solve the exercise we can also create a table where the days of the week are displayed at the dates and we consider the cases according to the starting day of the year.

The shorter version of this is the following:

If the 13th of *January is a Friday,* then we have found a month where there is a Friday 13.

Between the 13th of January and the 13th of February $31 = 4 \cdot 7 + 3$ days pass, the days are shifted by 3, so the 13th of *February is a Friday, only if the 13th of January is a Tuesday.*

If it is not a leap-year, between the 13th of January and the 13th of March $31 + 28 = 8 \cdot 7 + 3$ days pass, so the 13th of March will be a Friday exactly when the 13th of February is a Friday, and if the 13th of January is a Tuesday.

Between the 13th of January and the 13th of April $31 + 28 + 31 = 12 \cdot 7 + 6$ days pass, the days are shifted by 6, so the 13th of April is a *Friday, if the 13th of January is a Saturday.*

Between the 13th of January and the 13th of May $31 + 28 + 31 + 30 = 17 \cdot 7 + 1$ days pass, the days are shifted by 1, so the 13th of *May becomes a Friday, if the 13th of January is a Thursday.*

Between the 13th of January and the 13th of June $31 + 28 + 31 + 30 + 31 = 21 \cdot 7 + 4$ days pass, the days are shifted by 4, so the 13th of *June is a Friday, when the 13th of January is a Monday.*

Between the 13th of January and the 13th of July $31 + 28 + 31 + 30 + 31 + 30 = 25 \cdot 7 + 6$ days pass, the days are shifted by 6, so the 13th of *July becomes a Friday, when the 13th of January is a Saturday.*

Between the 13th of January and the 13th of August $31 + 28 + 31 + 30 + 31 + 30 + 31 = 30 \cdot 7 + 2$ days pass, the days are shifted by 2, so the 13th of August is a *Friday, when the 13th of January is a Wednesday.*

Between the 13th of January and the 13th of September $31 + 28 +31 + 30 + 31 + 30 + 31 + 31 = 34 \cdot 7 + 5$ days pass, the days are shifted by 3, so the 13th of September becomes a *Friday, if the 13th of January is a Sunday.*

We do not have to go on with the computing process, as whatever day the 13th of January is, we can always find a month that contains a Friday 13.

As a conclusion, since 1582, the start of the Gregorian calendar, there was a Friday 13 in each year. At that time 11 days were left out beginning with the 4th of October, to correct the Julian's calendar. There every forth year was a leap-year, but that results in an 11 minutes difference each year. To make up for this phenomenon, in the Gregorian calendar every year divisible by 100 is not a leap-year, with the exception of years divisible by 400, that are leap-years. The order of the months was the same in Julian's calendar as well, so Friday 13 always existed at that time also, even in 1582, as it had already cropped up in one of the months until September, and the days that were left out formed part of October.

5.5. Problem

Once in a June the date of three Thursdays was an odd number. What was the date on the last Sunday of that month?

Solution

As the date of three Thursdays is an odd number, the earliest Thursday is the 1st of June. If the 1st of June was a Thursday, then the 15th and the 29th of June are Thursdays too, in this case the last Sunday of the month is the 25th.

If the 3rd of June is Thursday, then the 17th of June is also Thursday, but 17 + 14 = 31, and June is 30 days long, that's why there are no more Thursdays with an odd date in June, so this is not a correct solution.

So, if the date of three Thursdays was an odd number in June, then the last Sunday of the month was on the 25th.

To solve the exercise we can also use the table of Problem 4, as it doesn't matter which day of the week the Problem is about.

5.6. Problem

What is the maximum number of months in a year that contain five Thursdays?

Solution

A month is at least 28, and at most 31 days long, so at least 4, and at most 5 Thursdays can be in a month.

A year is 365, a leap-year is 366 days long, this means 52 weeks plus 1 or 2 days, so there could be 52 or 53 Thursdays in a year. In a regular year there are 53 Thursdays, when the year starts with a Thursday, in a leap-year it can start either with a Wednesday, or with a Thursday and there will be 53 Thursdays.

If there are 4 Thursdays in each month, then there would be only 48 Thursdays, 5 more Thursdays remain, that could complete 5 months into ones with five Thursdays.

As a conclusion, there are at most 5 months in a year with 5 Thursdays.

To solve the problem we can also use the table of Problem 4, as it doesn't matter which day of the week the Problem is about.

5.7. Problem

One of Andrew and Ben was born in 1962, the other in 1963 or in 1964, but we do not know who was born when. Both of them have a watch with a face, but none is punctual enough. Ben's watch is ten seconds ahead in each hour, Andrews is ten seconds behind. In a January day both watches are set at twelve o'clock a.m. sharp. „Did you think of the fact", said Ben, „that the next time our watches show exactly the same time is only in March, on your next birthday – when you become 21 years old?"

„That's right", Andrew answered. Who is older, Andrew or Ben?

Solution

There are seemingly not much connection between the data of the problem and the question. Let's see the facts resulting from the data!

Let's compute first the number of days that will pass until the two watches show the same time again.

This will happen when the difference between them is 12 hours. As one of the watches is as much behind as the other is ahead, this will happen when Andrew's watch is 6 hours behind and Ben's watch is 6 hours ahead; it means, both of the watches will show 6 o'clock (though not correctly).

Andrew's watch	is 10 seconds behind in every hour,
	is behind by 60 s = 1 minute in 6 hours,
	is behind by 4 minutes in 24 hours = 1 day,
	is behind by 1 hour in 15 days,
	is behind by 6 hours in 90 days.

(The same could be computed more easily: 6 hours = 21600 seconds, so the watch will be behind with 6 hours in 21600 : 10 = 2160 hours. This will happen in 2160 : 24 = 90 days.)

So exactly 90 days later will the two watches show the same time.

As the earliest date they could set their watches was the 1st of January, and as in January there are 30 other days, March is 31 days long, the 90th day is in March only if February is 29 days long, i.e. it is a leap-year. This way we have found the missing link between the data and the question! As Andrew was born either in 1962, 1963 or 1964, and he becomes 21 years old either in 1983, 1984 or 1985, and his 21st birthday is in a leap-year, only 1984 can be the solution from the list. So Andrew was bom in 1963 and Ben was born in 1962, i.e. Ben is older than Andrew.

5.8. Problem

Choose a 3x3 square from the numbers on a sheet of a wall calendar. If you tell me the smallest number in the square, I tell you the sum of the numbers in the square. Try to compute it yourself as fast as possible!

Monday	*Tuesday*	*Wednesday*	*Thursday*	*Friday*	*Saturday*	*Sunday*
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30		

Solution

If we give this exercise to pupils it is worth leaving time for working out strategies, then make the pupils compete. The one who is the fastest in obtaining the result wins.

The exercise is suitable for the development of problem solving as every method that makes computing easier counts as a different result.

Many pupils realise how it is possible to add up the numbers in a row or in a column fast, as the sum of numbers in a row is three times the middle element of the row (or column).

Some also realise, that the sum of the second row is greater by 21 than the sum of the first row, and the sum of the third row is greater by 42 than the sum of the first row, so the sum of all the numbers is greater by 63 than three times the sum of the first row.

Following a similar track of thought, the sum of the second column is greater by 3 than the sum of the first one, the sum of the third column is greater by 6, so the sum of all the numbers is greater by 9 than three times the sum of the first column.

We get a faster result if we realise, than the sum of all the numbers is three times the sum of the middle row (or column).

Some notice, that the sum of numbers in the square is three times the sum of numbers in the diagonal of the square, regardless of the diagonal we choose.

It is only one more step following the last two observations to realise, that the sum of the middle row (column) and the sum of the diagonal is three times the middle element of the square, so the sum of the numbers in the square is nine times the middle element of the square. This way we reached at the fastest way of computing, as the middle element of the square is greater by 8 than the smallest element.

Note: the fastest way of multiplying a number with 9 fast, is to multiply the number by 10 and subtract 9 from the result.

Many pupils get the result by using a general formula to depict the numbers in the square. By denoting the smallest number with an X, the square is:

X	*X+1*	*X+2*
X+7	*X+8*	*X+9*
X+14	*X+15*	*X+16*

The sum of all the numbers in the square is $9X + 72 = 9 \cdot (X + 8)$

References

DUNN, A. (1980): Mathematical bufflers. Dover

GARDNER, M. (1992): Mathematical circus. MAA-

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6. Gold on a scale (Tamás Árki)

Our aim with these problems is to develop children's logic and problem-solving ability. The task series consists of 10 tasks built on one another.

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6.1. Problem

One day the captain of the pirates noticed, that his pirates have drank all he rum. As rum is a rather expensive drink, the captain wanted to punish his pirates. He said: „I don't know who drank all the rum, that's why I will punish everybody. Each of you has to give me 1 golden coin today!" And so they did; the captain got one-one coins from all his 27 men. Later he noticed, that one of the golden coins was a false one, and lighter than the other coins. The captain has a Roman balance at his disposal. Make as few scaling as possible to find the false coin! How many scaling are needed?

Solution

We will show that three scaling are enough to find the false coin. Divide the coins into 3 equal heaps, resulting in 9 coins in a heap. The heap containing the false coin is lighter than the other two. With the first scaling we look for the heap that is lighter than the other two; s0o let's put 1–1 heap into the two dishes of the scale. If the two dishes are in balance, then the false coin is in the third heap, while if the scale is not in balance, we should go on with our search in the lighter heap. (See figures 1 and 2)

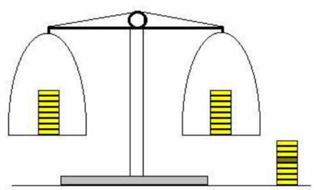

Figure 1
The two dishes are in balance, the false coin is in the third heap

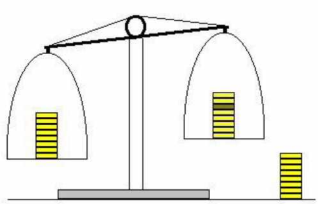

Figure 2
The dishes are not in balance, the false coin is in the lighter dish

We should choose only from 9 coins before the second scaling. Divide these 9 coins into 3 equal heaps; each heap will contain 3 coins. Following the previous method, we can choose the heap containing the false coin by one scaling. We should choose only among 3 coins at the last scaling. Put 1–1 coins into the dishes. If the scale is not in balance, then the lighter one is the false coin, if the scale is in balance, then the third coin (which is not on the scale) is the false one. (We have to start this problem with fewer coins (3, 9) for pupils.)

6.2. Problem

Solve the above problem if the number of pirates is

a) 81,
b) 54.

What is the least number of scaling to find the false coin?

Solution

a) If we divide the 81 coins into 3 equal heaps, one heap will contain 27 coins. We can tell on the basis of one scaling which heap contains the false coin. From the remaining 27 coins we can chose the false one in 3 scaling (see above). So, if we have 81 golden coins, 4 scaling are enough.
b) As $54=3\cdot 18$, we can reduce the number of suspicious coins to 18 with the first scaling. With the second scaling we choose 6 coins among which the false one is. Dividing the 6 coins into 3 parts there will be two coins in one group. With the third scaling we look for that pair which contains the false coin. In the last scaling we have 2 coins left to choose the lighter one from. It means, that in case of 54 coins 4 scaling are needed again.

6.3. Problem

In this exercise, each of the 27 pirates gives a golden piece of different weight to the captain. The mass of the golden pieces is almost the same, only a scale can show the weigh differences. The captain wants to keep the heaviest gold, and spend the rest on food and rum. Let's help the captain; find a solution to tell on the basis of the least number of scaling which gold is the heaviest. We can use only a Roman balance!

Solution

We work following a method similar to those of elimination tournaments. Put 1-1 golden pieces into the dishes, keep the heavier and "discard" the lighter one. Go on with the procedure until only one golden piece (the heaviest) remains. We get rid of 1 piece in every turn, so the number of essential scaling is 26.

6.4. Problem

The pirates give the captain 26 different pieces of gold again, the captain wants to keep the heaviest, but wants to give the lightest back, as the pirate have a day off today. Design a method that helps to state on the basis of the least number of scaling which one the lightest and the heaviest of the golden pieces is. We have a Roman balance as a help.

Solution

One solution is that following the above method we choose the heaviest gold (25 scaling), then with the same method choose the lightest one from the rest (24 scaling). The number of all scaling then is 25+24=49.

We will show that there is a better method. Create 13 pairs from the 26 golden pieces. Choose the heaviest one from each pairs with the help of one scaling; the golden pieces will be separated into two groups this way. One of the groups contains the "light ones", the other the "heavy ones". Both groups contain 13 pieces. It is also true, that the heaviest one is in the "heavy" group, and the lightest one is in the "light" group. Following the method presented above, we can find the lightest one in the "light" group with the help of 12 scaling, and the same method applies to the "heavy" group. The total number of scaling then: 13+12+12=37.

6.5. Problem

How can you alter the previous method in case of 27 golden pieces?

Solution

A possible alternative is the following. We should pair up the golden pieces again, but this time one is left out. Scale each of the 13 pairs and create the groups of "heavy" and "light" again (13 scaling).

Look for the lightest among the "light" group with the help of 12 scaling, then compare it with the one left out (this means 12+1=13 scaling). If the remaining (not paired) gold is lighter, than that is the lightest of all. Then choose the heaviest one from the "heavy" group with the help of 12 scaling. The number of scaling altogether is: 13+13+12=38.

If the remaining (not paired) gold is heavier than the chosen lightest piece, then the remaining gold is to be put to the "heavy" group, and we can choose from the 14 pieces the heaviest with the help of 13 scaling. In this case the total number of scaling is: 13+13+13=39.

In the next problem it is advisable to use a small amount first, as the problem is a bit difficult. A larger amount can also be given of course, depending on the abilities of the group.

6.6. Problem

Let's show that one can choose the two heaviest pieces of gold from among 8 golden pieces of different weight, using 9 scaling. Only a Roman balance can be used as a help!

Solution

Make 4 pairs out of the 8 golden pieces with the already known method, then separate the pieces into groups of "light" and "heavy" with 4 scaling. Make 2 pairs in the "heavy" group, then choose the heavier pieces of the pairs with two scaling. Choose the heaviest piece from the remaining 2 ones with 1 scaling; this is the heaviest of all. We have performed 4+2+1=7 scaling so far. Now we only have to find the second heaviest golden piece. Notice that the second heaviest piece could "drop out" only when it was compared to the heaviest one. The heaviest gold was scaled three times, i.e. it was compared to 3 other pieces. We can find the heaviest among these 3 golden pieces by 2 scaling. The total number of scaling is in fact: 7+2=9. If we draw a tree graph about this "tournament", we can see better the method.

6.7. Problem

In this exercise the captain was given 32 golden coins, among which one is false, and lighter than the others. Unfortunately the Roman balance used in the other exercises was stolen by the pirates. But the captain has found in the stockroom a one-dish chemist's scale that can be adjusted in a way that one calibration on the scale corresponds to the weight of a real golden coin. Design a method to find the false coin with as few scaling as possible!

Solution

Divide the gold into two equal heaps, containing 16-16 pieces. Placing one heap on the scale we can tell immediately which heap contains the false coin. If the needle reaches a whole calibration, the false coin is not in the dish (figure 3), while if the needle stays between two whole calibrations, the false golden piece is in the dish (figure 4). This way we can choose the group of 16 containing the false coin. The remaining 16 coins should be divided into two parts again. With one scaling we can decide which group of 8 contains the false coin. The procedure goes on; 4 and 2 coins will be in one group respectively in the next two scaling. There are only 2 coins to choose from in the last, fifth scaling. You can see that 5 scaling are enough to find the false piece of gold.

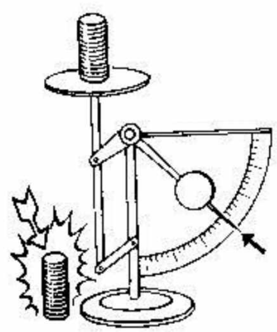

Figure 3
The needle reaches a whole calibration; the false gold is not in the dish.

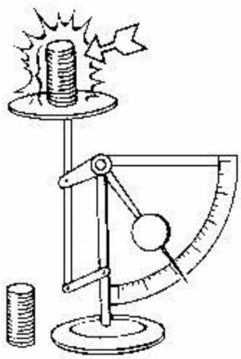

Figure 4
The needle stayed between two whole calibrations, the false coin is in the dish.

6.8. Problem

Once the pirates stole 10 chests of gold. Each of the chests was full of ingot gold. They also knew that there is false ingot gold or golden bars in one of the chests that are 1 gram lighter than the genuine ones. A genuine golden bar is 1 kilogram. The pirates thought they could find the false gold in 10 scaling. Then they thought they would put the captain to a test to find out whether he can solve the problem in less scaling. After listening to the pirates, the captain declared that he could choose the false gold in 1 scaling as well. The pirates did not believe him first. Do you believe him? What could be his track of thought?

Solution

The captain is right, 1 scaling is indeed enough to solve the problem. Number the chests from 1 to 10 then take as many golden bars from each as is indicated by the number of the chest. 1+2+...+9+10=55 bars were taken out of the chests this way, so if all the bars were genuine, they would weight 55 kilograms altogether. The real weigh though is as many grams less as many false bars there are among the selected ones; this number shows which chest the false bars were taken out.

6.9. Problem

The pirates have decided that they melt the golden bars and make smaller golden coins out of them. They have found a quite good Roman balance, and 3 and 5 gram weights, in enough numbers. How many grams could the pirates scale from the gold if they could put weights to both dishes? How many grams can they weigh if they have only 5 and 8 gram weighs?

Solution

Any grams of gold can be scaled in both cases. For this it is enough to show that 1 gram of gold can be scaled. Notice that in the first case $2\cdot 3=5+1$, i.e. if we put two 3 grams weighs to one dish, that can be balanced by a 5 grams weigh and 1 gram gold.

Similar solution can be used in the second case as well, by using the information $2\cdot 8=3\cdot 5+1$. This means that the two 8 grams weigh can be balanced by three 5 grams weigh and 1 gram of gold.

6.10. Problem

Is it true that any grams of gold can be scaled with the help of 2 and 6 grams weighs, if we have enough numbers of them? What is the case with 6 and 21 grams weighs?

Solution

It is not possible in either case to scale any numbers of gold. In case of the 2 and 6 grams weighs because $6=3\cdot 2$ the 6 grams weighs can be “changed” to three 2 grams weighs. So only the multiples of 2 can be weighed.

In the second case because $6=2\cdot 3$ and $21=7\cdot 3$ the 6 and the 21 grams weighs can be “replaced” by 3 grams weighs. So amounts that are not the multiples of 3 can not be weighed by 6 and 21 grams weighs.

What we can learn from last two exercises is that we can weigh any grams with the help of k and l gram weighs if and only if $(k, l)=1$.

7. Geometric dissection of Polygons (Tamás Árki)

7.1. Tangram-game

The elements of the Tangram-set are presented on Figure 1. Lay out these plain figures with the help of the elements of the Polydron, or cut them out from a chequered paper!

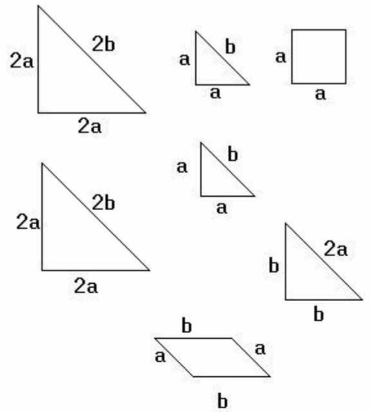

Figure 1

Try to form the figures on Figure 2, using the above elements, without any overlap! You have to use every element of the Tangram! A possible solution of the task is displayed on Figure 3.

Try to find the common characteristics of the figures! Examine the number of the vertices, perimeter and area! What properties are common of these plain figures? The proper answer is: the *area*.

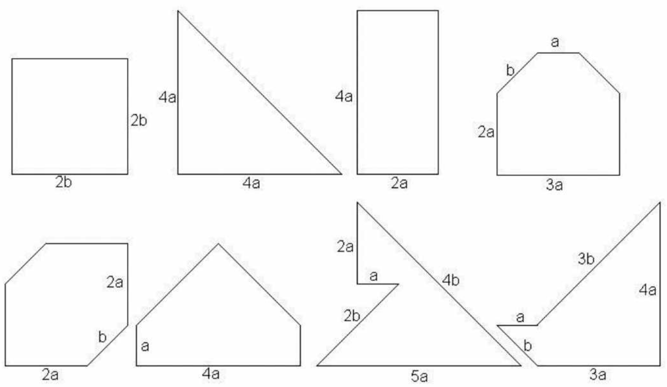

Figure 2

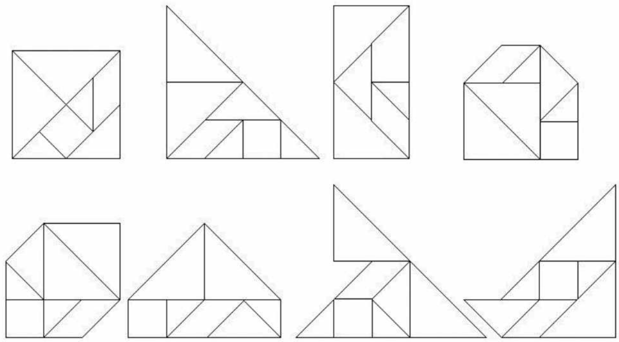

Figure 3

7.2. The question of geometric dissection of triangles of the same area

a) Use matches to create a triangle with sides of two matches long! With three more matches, connect the midpoints of the sides! The triangle is divided to four congruent triangles. Create a parallelogram by shifting two matches!

b) Cut an arbitrary triangle out of cardboard paper! On the basis of the solution ideas of the previous problem, dissect the triangle to make a parallelogram with the base equal to the base of the triangle!

You can trace the solution on Figure 4. F_1 and F_2 are midpoints on the sides of the ABC triangle. Revolve the triangle F_1F_2C around F_2 with 180°! The lines corresponded to sections F_1C and F_1F_2 are $F_1'B$ and $F_1'F_2$, respectively. That is why $F_1'B = F_1C = AF_1$, and $F_1'F_2 = F_1F_2$. As the middle line is parallel to and half as long as the corresponding side of the triangle, $F_1F_1' = AB$, and hence the length of opposite sides of the $ABF_1'F_1$ quadrangle is equal, Note that the base of this parallelogram equals the base of the AB triangle.

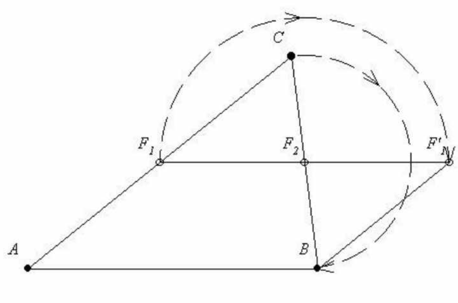

Figure 4

c) Using only one cut, dissect an arbitrary parallelogram into a rectangle with the base equal to the base of the parallelogram!

„Cut" a right-angled triangle along that altitude of the ABCD parallelogram, which goes through its vertex D, then shift the obtained ATD triangle by vector $\overrightarrow{AB}$. We get a $TT'CD$ rectangle of equal base with the triangle (see. Figure 5). The outlined procedure cannot always be applied in the present form. It is also possible, that T foot-end of the altitude line starting from point D does not lie on section AB, so the obtained triangle ATD is not part of the parallelogram. In this case, we do the cutting starting from another, suitable vertex of the parallelogram.

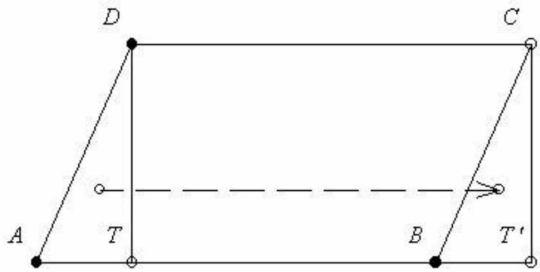

Figure 5

d) How could you dissect a triangle and rearrange it into a rectangle?
According to point b) rearrange it into a parallelogram first, and then according to point c), rearrange the resulted parallelogram into a rectangle.

e) Cut a trapezium out of a cardboard paper! Dissect and rearrange it with the help of one cut into a parallelogram of equal area! Try to apply the ideas presented in the previous problems!

We use the track of thought of point b), presented on Figure 6. Cut the ABCD trapezium into two along the F_1F_2 middle line, then revolve one of the resulted two trapeziums (trapezium F_1F_2CD in the figure) around point F_2 by 180°. Applying that the middle line of the trapezium is parallel to the bases and its length equals their arithmetic mean, and by applying the track of thought used in point b), we can see, that the quadrangle $AD'F_1'F_1$ is a parallelogram. According to the methodology of point c), the obtained parallelogram can be dissected and rearranged into a rectangle, so every trapezium can be dissected and rearranged not only into a parallelogram, but into a rectangle, as well.

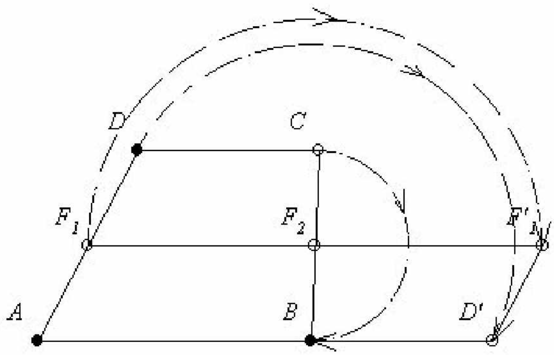

Figure 6

f) Cut out a convex quadrangle! Cut it into two parts along its middle lines, corresponding to the previous examples! (Middle line is a section connecting the middle points of opposite sides.) Try to create a parallelogram out of the pieces!

The solution is shown in Figures 7 and 8. First, we revolve parts marked by 1 and 4 around points Q and S by 180°, next to parts marked by 2 and 3.

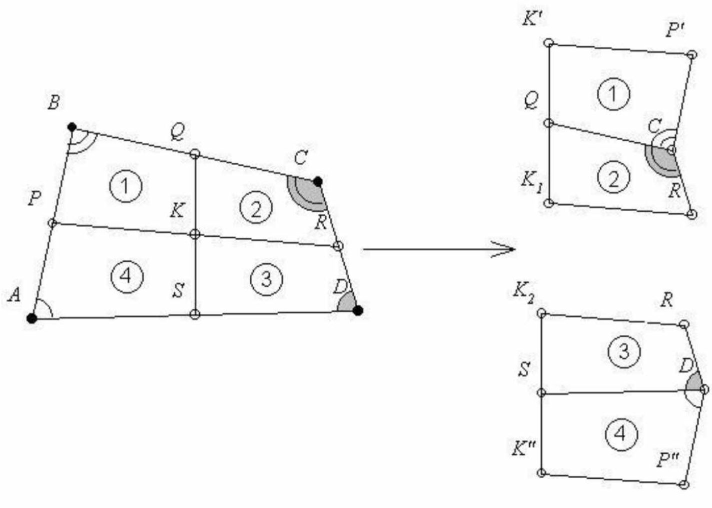

Figure 7

Then, using a suitable revolving, we can create a parallelogram out of the parts. This is shown in Figure 8.

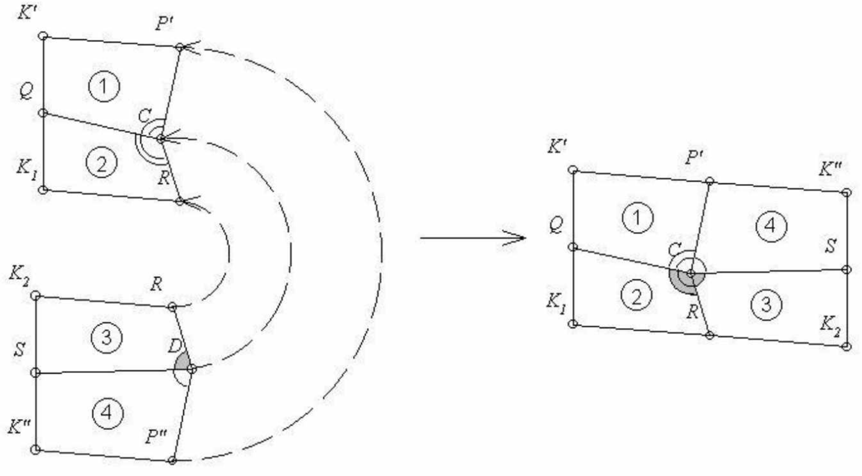

Figure 8

g) Try to solve the previous task in a way, that you cut the quadrangle along one of its diagonals instead of its middle lines!

In this case, we cut the quadrangle into two triangles of the same base. They can be dissected and rearranged into rectangles of the same base according to point d). Then, by putting them next to each other, we obtain the solution of the problem.

h) Dissect a concave quadrangle into a rectangle!

Along the diagonal starting from the vertex by the concave angle, we dissect the quadrangle into two triangles of the same base, then we dissect the triangles into rectangles, finally we join the two rectangles.
We can summarise all that was shown so far in one theorem.

Theorem
Any triangle and any quadrangle can be dissected into a rectangle.

i) Cut out a rectangle the sides are 5 cm and 10,5 cm, then cut out an other one, the sides are 7 cm and 7,5 cm! Use as few moves as possible to dissect one into the other!

We show that any two rectangles of equal area can be dissected into each other. Place the rectangles ABCD and AEFG of equal area as it is shown in Figure 9. (AB=c, AD=d, AE=b, AG=a), then "cut" the rectangle ABCD along the DE line! The rectangle disintegrates into the pentagon ABQPG (that is also present in rectangle AEFG), and into triangles GPD and DCQ. We show that the triangle GPD is congruent with the triangle BEQ, and the triangle DCQ is congruent with triangle PFE, where from the possibility of dissecting into each other already follows. It immediately follows from the construction that triangle GPD is similar to triangle BEQ, and triangle DCQ is similar to triangle PFE. Moreover, as we know about the area of the two rectangles that they are equal, it is enough to show the congruency of one of the triangle pairs. Applying the similarity of triangles ADE and BQE, it follows that $BQ = \frac{d(b-c)}{b}$, and we obtain from $dc = ab$ that $BQ = d - a$. Hence, BQ=DG, and it indeed means that triangle GPD is congruent with triangle BEQ, and triangle DCQ is congruent with triangle PFE.

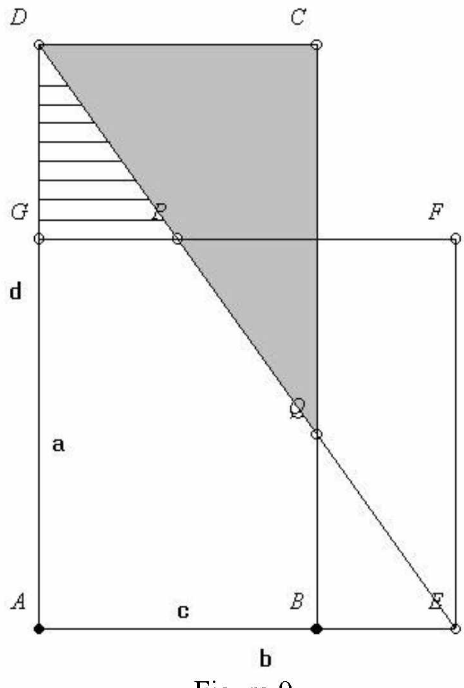

Figure 9

Figure 10 shows the way how the rectangle ABCD is rearranged into rectangle AEFG after the dissection. By a closer look at the figure, we can realise that our method works only in the case when point Q is an internal (maybe a limit-) point of rectangle AEFG. This applies when (using the signs of the figure) $0 \le d - a \le a$, that is $1 \le \frac{d}{a} \le 2$. If it does not apply, cut the rectangle ABCD along its shorter middle line, and join the two parts. Using these steps (might be repeatedly), we can obtain a case for which it holds.

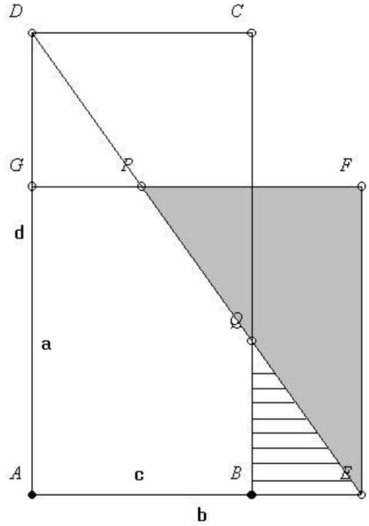

Figure 10

It follows from the previous dissection, that every rectangle can be dissected into a square.

j) There are two polygons of equal area. What are the steps to be used to dissect and rearrange one into the other?

   First we dissect one of the polygons into triangles, and then dissect each of the triangles into rectangles. Generally, it is not possible to create a rectangle out of these rectangles at once, as it is not ensured that they have sides of equal length. That is why we should first dissect each of the rectangles into rectangles of the same base, from which we can already assemble one rectangle. We can dissect this rectangle applying the above-presented method into a square. The other polygon can similarly be dissected into a square congruent with the obtained square. Hence, we have proved a theorem generally known as Bolyai–Gerwien theorem in the literature. Farkas Bolyai and P. Gerwien published the theorem independently in 1832, but W. Wallace has already published it in 1807.

Theorem

Every polygons of the same area can be dissected (by strait cuts) into each other.

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8. Regular Polyhedra (Lurdes Serrazina)

8.1. POLYHEDRA –SOME BASIC CONCEPTS

There is no more challenging problem when writing a book about the history of polyhedra than deciding what is meant by the term 'polyhedron'. A glance at the figures in this book will give you some idea of the variety of objects that have been described as polyhedra. Trying to make a catch-all definition is impossible as different writers have applied the same term to several different ideas, some of which are mutually exclusive. At the most elementary level we can ask whether a polyhedron is a solid object or a hollow surface. The answers to such questions depend to a large extent on the period in which the geometricians lived and the problems that they studied. To a classical Greek geometrician a polyhedron was solid. Over the past 200 years it has become more convenient to think of polyhedra as surfaces. Today, some geometricians regard polyhedra as frameworks.

It has been said that the only thing all polyhedra have in common is the name. However, there is some common ground to be found. Their most obvious property is that they are made of (or bounded by) polygons. This fundamental property constituted a definition of 'polyhedron' for many centuries, even if wasn't stated explicitly. As we shall see, such an open-ended definition can be interpreted in many ways. It does not supply any restriction on how the polygons are to be put together or on what kinds of polygon we can use. This ambiguity has been extremely fruitful, allowing the term to evolve in several directions and leading to the study of different kinds of polyhedral objects. Because of this we shall leave 'polyhedron' as this vaguely defined term. (…)

- Each polygon is called a face of the polyhedron;
- A line segment along which two faces come together is called an edge;
- A point where several edges and faces come together is called a vertex.

We shall make a distinction between the component parts of a polygon and those of a polyhedron. A polygon has sides and angles (or vertices), while a polyhedron has faces, edges and vertices. Each edge is formed by the sides of two faces. Two faces that form an edge are said to be adjacent.

There are also various types of angle in a polyhedron:

- The angle in the corner of a polygonal face is called a plane angle;
- In a solid polyhedron, the region of the polyhedron near a vertex is called a solid angle. It is the part of a corner that is bounded by three or more plane angles;
- The angle between two adjacent faces is called a dihedral angle.

(Adapted from Polyhedra, pp.12–14, Peter R. Cromwell, Cambridge University Press, 1997)

CLASSIFICATION OF POLYHEDRA

A family of polyhedra is basically a set of polyhedra organised according to their characteristics.

The characteristics of each family can be discussed quickly, through observation, with the possibility of a rapid comparison of elements from different families and the finding of counter-examples.

Here is a list of some families of polyhedra:

PLATONIC SOLIDS

Polyhedra whose faces are regular polygons, congruent (geometrically equal) and arranged in such a way that the same number of edges meet at each vertex (the polyhedral angles are congruent).

ARCHIMEDEAN SOLIDS

Archimedean solids are convex polyhedra whose faces are all regular polygons, but not all congruent (as opposed to the five regular polyhedra or Platonic solids, whose faces are all regular congruent polygons), and whose edges and vertices are all congruent. Thus, a solid Archimedean must necessarily include two or more different regular polygons, with the same polygons occurring around each vertex in the same sequence for example: hexagon/hexagon/triangle in the truncated tetrahedron. These polyhedra are the result of truncations made in the vertices of the Platonic solids.

These solids, studied by Archimedes (287–212 B.C.) are also known as semi-regular polyhedra.

CONVEX DELTAHEDRA

Convex deltahedra are polyhedra whose faces are equilateral triangles. This is a very interesting family because it has an infinite number of elements and a regularity of construction with a totally unexpected fault.

Deltahedra are members of the Platonic and Johnson families of polyhedra.

There are other polyhedra, besides the Platonic solids, which are bounded by regular polygons. For example, convex deltahedra are all bounded by equilateral triangles. This family is composed of eight convex polyhedra and include the tetrahedron, the octahedron and the icosahedron. A polyhedron is convex if we can stand it on a table on each of its faces.

The Delthedron Family

Deltahedra are all the polyhedra whose faces are equilateral triangles. The designation derives from the agglutination of the terms delta and hedron - delta by analogy between the form of the equilateral triangle and the Greek capital letter of the same name; hedron because in Greek it means plane face:

delta + hedron = triangle + plane face

It was demonstrated in this century that there are only eight convex deltahedra.

Name	*n⁰ of faces*	*n⁰ of vertices*	*n⁰ of edges*	*Meaning of name*
Tetrahedron	4	4	6	tetra+hedron (4 plane triangular faces)
Triangular Bipyramid	6	5	9	bi+pyramid (2 regular pyramids with triangular base)
Octahedron	8	6	12	Octa+hedron (8 plane triangularfaces)
Pentagonal Bipyramide	10	7	15	Bi+pyramide (2 regular pyramids with pentagonal bases)
Dodecadeltahedron	12	8	18	do+deca+delta+hedron (2+10 plane triangular faces)
Tetradecadeltahedron	14	9	21	tetra+deca+delta+hedron (4+10 plane triangular faces)
Hexadecadeltahedron	16	10	24	hexa+deca+delta+hedron (6+10 plane triangular faces)
Icosahedron	20	12	30	icosa+hedron (20 plane triangular faces)

The name deltahedron was applied by Martín Cundy to any polyhedron bounded by equilateral triangles. The capital Greek letter of the same name reminds us of these triangles.

Two of the five non-regular deltahedra are bipyramids. Another is formed by joining three square-based pyramids to the faces of a triangular prism. A fourth is obtained by starting with a square-based antiprism and raising pyramids on both the square faces. The names given to these two deltahedra are attributed to Norman Johnson, who devised a designation for polyhedra with regular faces, as in the case of Archimedean polyhedra. The remaining solid is not so easy to describe. It has 12 faces and is often known as the *siamese dodecahedron*, a name coined by H. S. M. Coxeter. Johnson called it the *snub disphenoid*.

"Where can we find deltahedra?"

Nº of faces	Designation (POLY)	Family name
4	Tetrahedron	Platonic Solids
6	Triangular Dipyramide	Johnson Solids (J12)
8	Octahedron	Platonic Solids
10	Pentagonal Dipyramide	Johnson Solids (J13)
12	Snub Disphenoid	Johnson Solids (J84)
14	Triaugmented Prism Triangular	Johnson Solids (J51)
16	Gyroelongated Square Dipyramide	Johnson Solids (J17)
20	Icosahedron	Platonic Solids

PRISMS AND ANTIPRISMS

Prisms and Antiprisms have two bases, which may be any regular polygon; prisms have rectangular lateral faces, and antiprisms have triangular lateral faces.

The designation of the polyhedron depends on the polygon of the bases.

A prism is a polyhedron with two congruent, parallel faces – the bases – and whose remaining faces – lateral faces – are parallelograms. Prisms are designated according to the form of their bases: a prism whose base is a triangle is a triangular prism; a prism whose base is hexagonal is a hexagonal prism, etc. Prisms may be straight, if the lateral faces are perpendicular to the bases; otherwise they are known as oblique. A regular prism is a straight prism whose bases are regular polygons. If we rotate one of the bases of a prism until each vertex of one base is facing one side of the other base and join the vertices with triangles, we get an antiprism.

Prisms with regular lateral faces are, according to the definition made earlier, Archimedean solids. However, prisms and antiprisms (of which there are an infinite number) are not generally included in the family of Archimedean solids.

In prisms there are at least two congruent faces, placed on parallel planes designated as bases. Provided it is not a quadrilateral, any pair of polygons in this case will be the base. The lateral faces should always be parallelograms. If the lateral faces are all rectangles, the prism is called a straight prism; if any of the faces is a non-rectangular parallelogram, the prism is called an oblique prism.

The prism is called a regular prism if it is straight and the base is a regular polygon.

And if the faces are quadrilaterals?

If there is a set of adjoining rectangular faces, it will be a straight prism, whatever the base quadrilateral.

Prisms with special designations

If the lateral faces are rectangular, and the polygon of the base is a square, the prism is usually known as a *quadrangular prism,* and therefore a regular prism. In this case, but when the lateral faces are all square, what we have is a special regular quadrangular prism, or cube.

Parallelepipeds

If all the faces are parallelograms, necessarily parallel to each other, any pair of faces can be the base. In this case we are talking of a *parallelepiped* (or *non-rectangular parallelepiped*). Parallelepipeds of this type are straight, if there is a set of lateral faces that are rectangles.

If all the faces are rectangles, this prism is known as a rectangular parallelepiped and is always a straight prism; in this case, and if it is regular, it is either a quadrangular prism or a cube.

Antiprisms

In antiprisms there are two types of faces that are the crucial elements: the lateral faces and the bases.

The bases are congruent polygons, belonging to parallel planes, in which one of the bases is rotated so that each vertex of one base is facing one side of the other.

If, after rotation, we join the vertices of the bases to form triangles, we get lateral faces that will always be triangles.

If the bases are regular polygons and the rotation is done in such a way that the vertex of one of the bases corresponds to half the opposite side of the other base, the triangles of the lateral faces will always be isosceles triangles.

Pyramid

Pyramids are polyhedra bounded by a polygon (base of the pyramid) and by triangles (lateral faces). In pyramids, all but one of the faces will be triangular. The face that may not be a triangle is the base, and the others are lateral faces that meet at a single point called the vertex of the pyramid. The designation of the pyramid depends on the polygon of the base. In triangular or tetrahedral pyramids (polyhedra with four faces), any face may be the base; the choice of face for the base should conform with the other two criteria for analysis - whether the pyramid is straight and/or regular.

If the perpendicular to the base, taken from the vertex, passes through the centre of the base, the pyramid is *straight*; if not, it is *oblique.*

If the pyramid has a regular polygon as its base, and if the lateral edges are all equal, the pyramid is *regular.*

Bipyramid

Bipyramids are obtained when two pyramids with congruent bases are juxtaposed in such a way that their bases fit perfectly together. In this case, the base of the bipyramid will be the base of the pyramids from which it is fomed. The “base” is not a face of the polyhedron. The polygon of the “base”, thus defined, belongs to a single plane, which may or may not be a plane in symmetry.

The faces of a bipyramid are all triangles.

Duality among Polyhedra

The first systematic study of the duality of polyhedra is usually attributed to E. C. Catalan. It therefore comes as no surprise that if we look for the dual of each of the Archimedean polyhedra, we find a new family of precisely that name: the Catalan polyhedra.

One way of obtaining a dual polyhedron from an Archimedean polyhedron or a Platonic polyhedron is to take the centre of each of the faces and join it, with edges, to the centres of the adjoining faces. The new polyhedron will be the dual of the old one.

CATALAN SOLIDS

Catalan solids are the dual polyhedra from Archimedean solids.

DIPYRAMIDS AND DELTOHEDRA

Dipyramids are the duals of prisms.

Deltohedra are the duals of antiprisms.

JOHNSON SOLIDS

An interesting problem is trying to locate the complete set of convex polyhedra with faces that are regular polygons. This set includes the Platonic solids, the Archimedean solids and some prisms and antiprisms as specific cases. All convex polyhedra whose faces are regular polygons, and which are not Platonic or Archimedean solids, prisms or antiprisms, are the so-called Johnson solids. Finding all the Johnson solids, 92 in all, is a daunting task. Norman Johnson (1966) published a complete list of these 92 solids, gave them names and conjectured that there could not be any more. Zagaller (1969) proved that the list was, in fact, complete.

There are also other solids, such as the Starred Solids and the Kepler-Poinsot Solids.

There is a shareware programme on the Internet called Poly which allows you to view these solids and obtain their plans for future construction. You can access this site on http://www.peda.com/poly/

Further information may be obtained at the following sites:
In Portuguese:
http://www.fc.up.pt/atractor/mat/duais.html
In English:
http://www.georgehart.com/virtual-polyhedra/naming.html
http://mathworld.wolfram.com/Polyhedron.html
http://forum.swarthmore.edu/alejandre/workshops/unit14.html
http://www.inf.ethz.ch/department/TI/rm/unipoly/index.html
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http://www.georgehart.com/virtual-polyhedra/prisms-info.html
Duality and Viewing of Catalan Solids:
http://www.georgehart.com/virtual-polyhedra/duality.html
http://www.georgehart.com/virtual-polyhedra/archimedean-duals-info.html
Johnson Solids: http://www.georgehart.com/virtual-polyhedra/johnson-info.html

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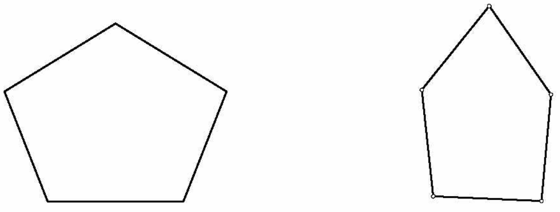
8.2. Activities on regular polyhedra

Activity 1

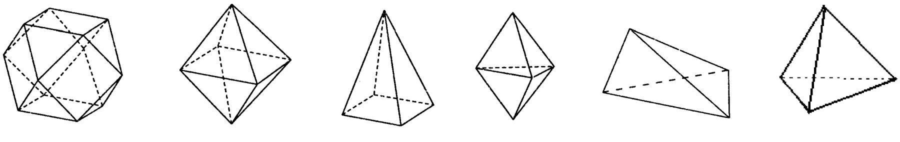
Look at these polygons:



They are both pentagons, but the one on the left is said to be *regular,* because all the sides and all the internal angles are congruent. In the polygon on the right, the sides are congruent but the internal angles are not. This is why Euclides, a Greek mathematician who lived 2.300 years ago, referred not to regular polygons but to equilateral (equal sides) and equiangular (equal angles) polygons.

Besides polygons, there are also three-dimensional polyhedra.

Look carefully at the five polyhedra below, construct them using the polydrons and try to see how the word *regular* could also be applied to polyhedra. Note: With the exception of C and E, the other polyhedra should be constructed only with regular polygons.



A **B** **C** **D** **E** **F**

Look at the kinds of vertices and faces in these polyhedra. By analysing the data from your observation, is it possible to designate any of these polyhedra as regular?

a) The group should note their observations of each of the polyhedra and justify their choice as to whether they are regular or not;

b) The group should work out a definition for a regular polyhedron.

Activity 2

Bearing in mind your definition of a regular polyhedron:

– Work out how many regular polyhedra it is possible to construct;
– Try to work out a justification for the existence of only a limited number of regular polyhedra.

Suggested task:

– *Begin by constructing regular polyhedra only with equilateral triangles and with each face constructed with only one piece from the polydrons;*
– *Do the same with squares, pentagons, etc.*

r e p o r t

Group	Class	Date

Activity 3

A polyhedron is said to be regular *when its faces are regular polygons, congruent (geometrically equal) and arranged in such a way that the same number of edges meet at each vertex.*

A regular polyhedron must satisfy *three* criteria.

Activity 3.1

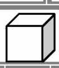
Construct polyhedra that satisfy the following criteria:

a) They have faces that are not regular polygons (the other two criteria are retained);
b) They have faces that are not congruent (the other two criteria are retained);
c) They have vertices that are not of the same type - order of the vertex (the other two criteria are retained).

Activity 3.2

Construct polyhedra that satisfy one, and only one, of the criteria as defined.

The group should organise a table, of the type shown below, to record the features of the constructed polyhedra:

Conditions F=	**V=**	**FR**	**Polyhedron Designation**	**Representation**	**Faces nº**	**Faces polygon**	**Vertices nº**	**Vertices order**	**Edges nº**	**Polyhedron Description**
V	V	**V**	CUBE		**6**	QUADRADO	**8**	**3**	**12**	Hexahedron with regular faces, all congruent, parallel two by two.

"**F=**" congruent faces "**V=**" congruent vertices "**FR**" regular faces	*Note:* *The number of edges meeting at a vertex is called the **order of a vertex** .*

Activity 3.3

Look at the data you obtained for each polyhedron. Is there any way of knowing the number of edges of a polyhedron without counting them, bearing in mind the number of faces and vertices?

Activity 4

POLY Programme on polyhedra that can be accessed through the Internet site: *www.peda.com*

1. **a)** Select some known polyhedra (e.g. Platonic) and analyse their plan and respective Schlegel diagram (last icon on the right of the tools window).
   What are the advantages and/or disadvantages of each of the situations?
   What should be borne in mind when counting the faces and edges by referring to the Schlegel diagram?

   b) Describe the following polyhedra, explaining the method used:
   The TRUNCATED TETRAHEDRON, of the Archimedean polyhedra family;
   The TRIAKIS TETRAHEDRON (J41), of the Catalan polyhedra family.
   You should bear in mind the number of faces, vertices and edges, the Euler relation, the type of faces and vertices and other relevant relations.

2. Describe each of the families of polyhedra shown in the POLY.
   You should refer in your description only to the basic elements of the respective family.
3. By establishing a classification of increasing demand, indicate the polyhedra (families or specific polyhedra) that fulfil the following requirements:
   a) Polyhedra with congruent faces;
   b) Polyhedra with all faces congruent and all regular polygons;
   c) Polyhedra with congruent faces and vertices and the faces of the polygons regular;
   d) Polyhedra with more than one type of face, all of them regular polygons;
   e) Polyhedra with more than one type of face, all of them regular polygons with congruent vertices.

4. The triangular dipyramid and the pentagonal dipyramid (J12 and J13) belong to the Johnson polyhedra.
   Should the quadrangular and the hexagonal dipyramid be included in this family of polyhedra? Why/why not?
   Are there any other dipyramids?

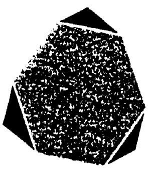
5. **Convex deltahedra** are polyhedra whose faces are equilateral triangles.
   "Where can you find deltahedra?"

6.

a) In the description of certain polyhedra, some of the faces play a fundamental role, bearing in mind, for example, relations of parallelism between the faces or the position of one face in relation to the others.
To which families of polyhedra does this apply? Why?

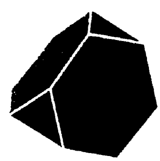
b) For each of the following families of polyhedra, prisms, antiprisms, pyramids and bipyramids, define a general expression to determine number of faces, edges and vertices in relation to the number of sides of the polygon of the base.

7. The *dual of a polyhedron* is the polyhedron formed by starting with the centre of each of the faces and joining, by edges, the centres of the adjacent faces of the original polyhedron.



If we consider, for example, the truncated tetrahedron, we can note that its dual should have 8 vertices that cannot all be congruent.
There will be 4 vertices, each the exit point for 6 edges.
And there will be 4 vertices, each the exit point for 3 edges.
This happens because in the truncated tetrahedron there are hexagonal and triangular faces.

Another feature of the dual of the truncated tetrahedron is that all its faces should be congruent triangles which are isosceles triangles. This is because all the vertices of the truncated tetrahedron are congruent of the type 3.6.6. The dual of the truncated tetrahedron is the *triakis tetrahedron.*



Using the method described, work out the characteristics of the duals of the Archimedean polyhedra and identify them from among the Catalan polyhedra. Make a record of your procedure and conclusions.

8. Classification and description: what distinguishes each of these activities?

9. Work out a method for counting the elements of a polyhedron by referring to the type of vertices and the number and type of faces.

Regular Polyhedra

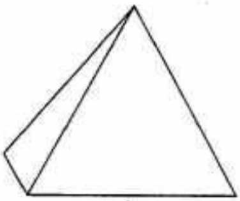
Polígono da face	Order of Vertices	F Nº of faces	A Nº of edges	V Nº of vertices	Name of polyhedron
TRIÂNGLE	3	4	6	4	TETRAHEDRON
					HEXAHEDRON OR CUBE
					DODECAHEDRON
					OCTAHEDRON
					ICOSAHEDRON

Note: the **Order of a Vertex** is the number of edges concurring on it.

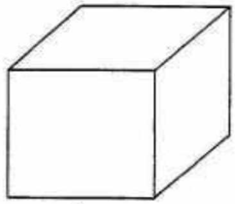
CUBE (EARTH) TETREAHEDRON (FIRE) DODECAHEDRON (UNIVERSE) ICOSAHEDRON (WATER) OCTAHEDRON (AIR)

THE NINE REGULAR POLYHEDRA

The five regular convex polyhedra: 1 - Tetrahedron; 2 - Cube; 3 - Octahedron; 4 - Dodecahedron; 5 - Icosahedron. These have been known since antiquity. It is to KEPLER (1571–1630) that we owe the discovery of the first regular concave polyhedron – the starred dodecahedron with regular faces shown in Fig. VI. In 1809, the dentist FRANCIS LOUIS POINSET (1777–1859) added three new regular, non-convex polyhedra to this list (VII, VIII & IX). Meanwhile it was CAUCHY who demonstrated that there only exist these nine regular polyhedra. It should be noted that each regular concave polyhedron is the result of prolonging the faces of a regular convex polyhedron, which serves as its nucleus, as can be seen in KEPLER's starred dodecahedron (VI), which is the result of the prolonging of the dodecahedron (IV).



I



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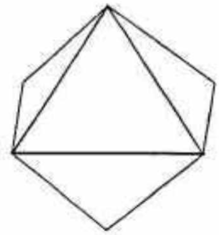

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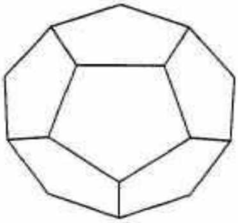

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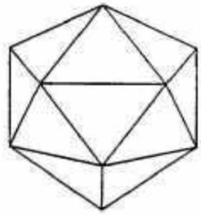

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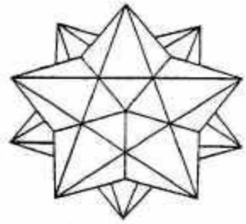
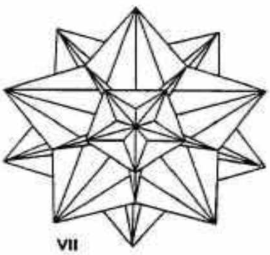
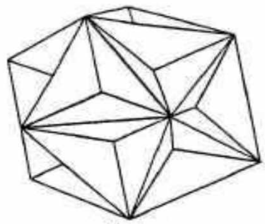


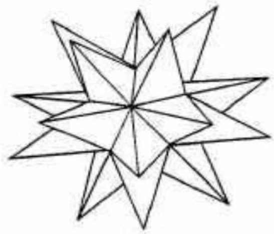
VI



VII



VIII



IX

Euclides' demonstration
that there can be no more than five regular polyhedra.

The research you have been working on was conducted originally by Greek mathematicians, several centuries before our own time. The mathematics books from those days have almost all been lost. But we still have one very famous book called *Elements,* written by the mathematician EUCLIDES, in which he demonstrates that there are only five regular polyhedra.

EUCLIDES was a researcher and teacher in Alexandria, a city which is today part of Egypt but which in those days – about 300 years before our own time – belonged to Greece. Apart from this, little is known about his life. EUCLIDES' *Elements* is in 13 volumes, the last two of which are devoted to spatial geometry. The theorem relating to regular polyhedra is the last in the thirteenth volume.

When you read EUCLIDES' text, note that:

- Euclides used *shapes* to refer to three-dimensional as well as to two-dimensional shapes;
- Instead of saying *regular polygons,* EUCLIDES says *equilateral* and *equiangular* polygons (Are they the same thing?);

A solid angle is the shape formed by a vertex and the faces concurrent on it.

Euclides' Text

I say moreover that *no other shape, besides the five mentioned, can be formed by equilateral and equiangular polygons equal to each other.*
A solid angle cannot be constructed with two triangles, or even two plane angles.
With three triangles you form the angle of the pyramid, with four the angle of the octahedron, and with five the angle of the icosahedron;
But a solid angle cannot be formed by six equilateral and equiangular triangles placed on the same point because, as the angle of the equilateral triangle is two-thirds of a right angle, the six will be equal to four right angles:
Which is impossible because any solid angle is formed by fewer than four right angles. For the same reason, neither can a solid angle be formed by more than six plane angles.
The angle of the cube is formed by three squares, but with four it is impossible to form a solid angle, because there would again be four right angles.
The angle of the dodecahedron is formed with three equilateral and equiangular pentagons; but with four it is impossible to form any solid angle because, since the angle of the equilateral pentagon is a right angle and one-fifth, the four angles will be greater than four right angles: which is impossible.
In the same way, no other solid angle cannot be formed by other polygonal shapes, for the same reasons of the absurd. Therefore, etc. . .

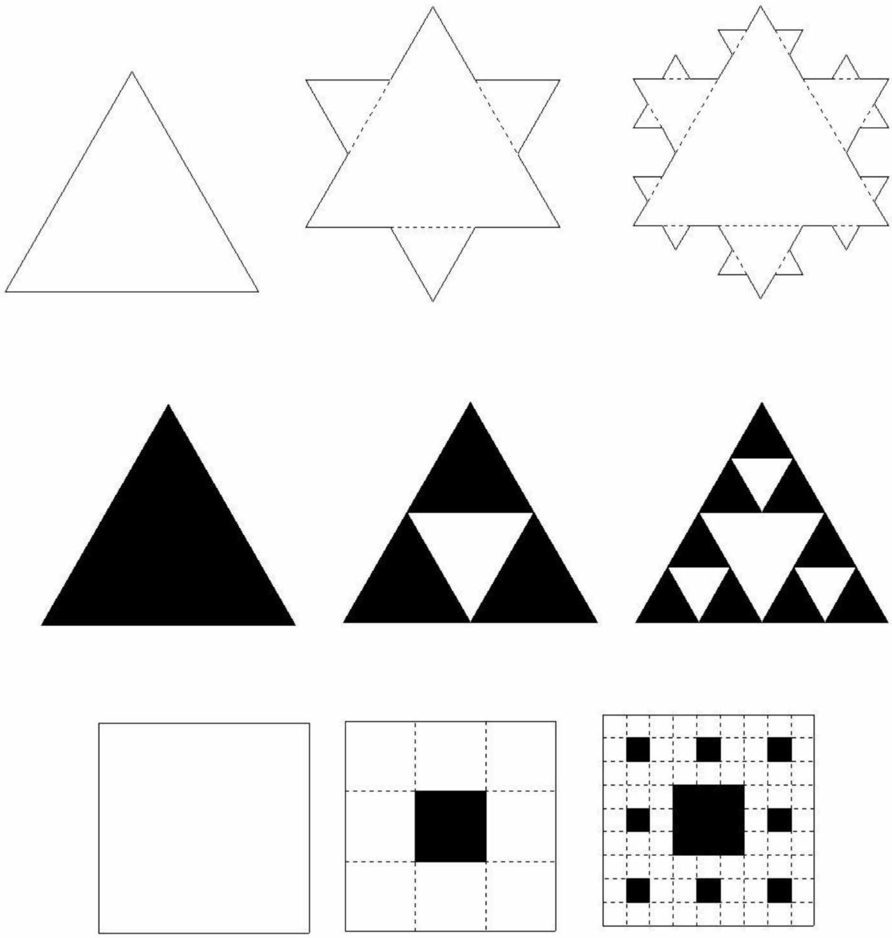
Elements, Book XIII, p.507–508.

9. Triangles, tetrahedrons, sequences (Klára Pintér)

The following problems develop spatial thinking through building solids. For this concrete manipulation, solids created out of paper, with the help of scissors and glue are extremely essential.

9.1. Problem

Draw the next two members of the sequence of the figures. Measure the circumference and the area of the objects! Count the circumference and the area of the objects if one side of the first figure is 1 unit. What can we say about member n of the sequence? What can we say about the bounded nature of the circumference and the area of the sequence?



Solution

We give the solution referring to the first sequence, the rest can be calculated similarly:

For the 1 unit side regular triangle:

The circumference: $K_0 = 3$

The area: $T_0 = \frac{1}{2} \cdot \frac{\sqrt{3}}{2}$

In case of the second figure, the length of each side is: $\frac{1}{3}$; the number of sides is: $3 \cdot 4$.

The circumference: $K_1 = \frac{1}{3} \cdot 3 \cdot 4$

The area: $T_1 = T_0 + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{3^2} \cdot 3$

In case of the third figure the length of each side is $\frac{1}{9}$; the number of sides is $3 \cdot 4 \cdot 4$.

The circumference: $K_2 = \frac{1}{9} \cdot 3 \cdot 4 \cdot 4$

The area: $T_2 = T_1 + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{9^2} \cdot 12$

We can observe, that after each trisection the number of sides becomes four times as much, the length of sides becomes one-third as much, so the circumference becomes 4/3 as much as the previous one, in case of member n: $K_n = 3 \cdot 4^n \cdot \frac{1}{3^n}$, that exceeds every limits if n increases, that is the circumference of the figures in the sequence is not limited bounded.

The area generally is:

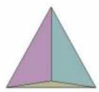
$$T_n = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{1}{3^2} \cdot 3 + \frac{\sqrt{3}}{4} \cdot \frac{1}{9^2} \cdot 3 \cdot 4 + \ldots + \frac{\sqrt{3}}{4} \cdot \frac{1}{\left(3^n\right)^2} \cdot 3 \cdot 4^{n-1} = \frac{\sqrt{3}}{4} \cdot \left(1 + \frac{3 \cdot \left(1 - \left(\frac{4}{9} \right)^n \right)}{5} \right),$$

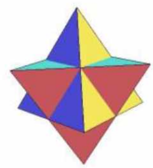
from which we can see the fact already suggested by the drawings, that the area of the objects in the sequence is bounded.

This way we obtain objects the circumference of which can be optionally large, while their area is limited bounded. The parts of the objects are similar to the whole object, these kinds of figures are called fractals, the dimensions of which are not necessarily integer. Research concerning fractals are a new branch of mathematics.

9.2. Problem

Create the first two members of the following sequence out of paper! Calculate their surface and volume, if one edge of the first tetrahedron is 1 unit long. What can we say about the limits of the surface and the volume of the members of the sequence? Place following members of the sequence into a cube! What can we state?





Solution

Altitude of the base of the 1 unit side triangle is: $m = \frac{\sqrt{3}}{2}$, the altitude of the solid is: $M = \sqrt{\frac{2}{3}}$, so the area of one face of it is: $T_{\text{lap}} = \frac{\sqrt{3}}{4}$, the surface of the tetrahedron is: $A_0 = 4 \cdot T_{\text{lap}} = \sqrt{3}$, and its volume is: $V_0 = \frac{1}{3} \cdot T_{\text{lap}} \cdot M = \frac{\sqrt{2}}{12}$.

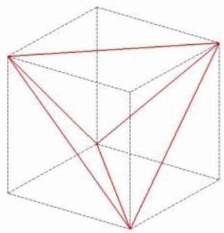
The area of the triangle faces after the first building is: $\frac{1}{4} \cdot \frac{\sqrt{3}}{4}$.

The number of bordering triangle faces is: 6·4=24.

The surface of the obtained solid is: $A_1 = 6 \cdot 4 \cdot \frac{1}{4} \cdot \frac{\sqrt{3}}{4}$, its volume is:

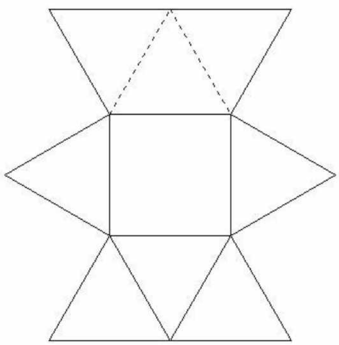
$V_1 = V_0 + 4 \cdot \frac{1}{8} \cdot V_0$.

The suface of the members of the sequence is not limited, while their volume is. Members following one another fill the cube more and more, that is shown by placing the regular tetrahedron into the cube:



9.3. Problem

Create the following solid net in two copies and assemble the solids! Build a tetrahedron by joining the two solids! What experiences can be obtain from this assembling?



Solution

As the finished tetrahedron can not have a square shaped face, the two solids are to be joined in a way that the square faces meet. If me assemble the two solids in a way, that the edges opposite to the square faces are parallel, we do not yet get a tetrahedron, but if we revolve them by 90°, these edges will be skew edges, the skew edges of the finished tetrahedron.

It is visible, that the originally folded two solids were the results of intersecting the plane parallel to the skew edges of the tetrahedron.

The intersection of the plane is a square, because the intersecting plane intersects the edges at their middle points, so we obtain two congruent solids.

9.4. Problem

Glue balls according to the figure (It can be created out of Plastic, tied together by tooth-picks.). Create a tetrahedron out of these four pieces! What experiences can we gain from this assembling? What pieces can be used to create a tetrahedron of five ball edge length similarly?

Solution

The two skew edges of the resulted tetrahedron are the two 4x1 pieces, as only an edge can be 4 balls long. If we fit the 3x2 piece along, then the other 3x2 piece crosswise on the 4x1 piece, then finally we fit the other 4x1 piece on this, we get the tetrahedron.

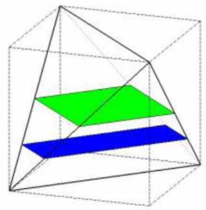
Just like in case of the above example, the difficulty here again lies in the fact that we have cut the tetrahedron with planes parallel to its two skew edges.

For the tetrahedron with an edge length of 5 we need two 1x5, two 2x4 and one 3x3 piece, and the way of construction is similar to the previous ones.

The problem can be further developed for example for plain-segments parallel to a plain of the tetrahedron as well. In this case for the given plain segments the number of balls is a triangle-number, the sum of these triangle-numbers is a tetrahedron-number. This direction leads towards figural numbers, which is very interesting in terms of experience and induction and it has several mathematical-history implications too.

9.5. Problem

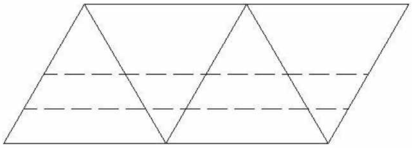
Let's build a tetrahedron, then wrap it up in a paper in a way that the paper should neatly fit the faces of the tetrahedron, and it is open along one edge of the tetrahedron only, and otherwise it is closed. The question is, is it possible to "use magic" and take the tetrahedron out of this "paper dress" without tearing the paper or squeeze the tetrahedron? What experiences can we gain from this experiment concerning the plain-segments of the tetrahedron that are parallel to the skew edges?



Solution

We can carefully push the tetrahedron out of the paper if we try to push the skew edge-pair of the open edge out. The circumference of the gap is the double of the tetrahedron's edge length, and that is continuously the area of the plain-segments parallel to the skew edges of the tetrahedron. So we guess the circumference of plain-segments parallel to the skew edges of the tetrahedron are constant: twice as long as the length of the edges of the tetrahedron.

If we draw the circumference of the plain-segment on the net of the tetrahedron we can see that the above guess is correct, as parts between the two opposite sides of the parallelogram that are parallel to the other side, are of equal length to the side of the parallelogram parallel to them.



Plain-segments parallel to the diverting edges of the tetrahedron are all rectangles. As the area of the square is the smallest from rectangles of equal circumference, the area of the square is minimal among plain-segments that are parallel to the skew edges of the tetrahedron.

It is visible on the basis of the above examples that the plain-segment that is parallel to the skew edges of the tetrahedron will be a square if the cutting plain cuts the proper edges of the tetrahedron in half.

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10. The secret of Harry Potter (Klára Pintér)

In the book, *Harry Potter and the Philosophers' Stone,* Evil One went in quest of the Philosophers' Stone that was hidden in the school of wizardry and protected by magic. Harry and his friends followed Evil One and arrived to a room with seven bottles of different shape, standing next to each other on a table. Purple fire flared up behind them at the door leading to the room, and there was a black fire on at the door leading to the next room. They discovered a scroll next to the bottles, with the following instructions:

Danger lies before you, while safety lies behind,
Two of us will help you, whichever you would find,
One among us seven will let you move ahead,
Another will transport the drinker back instead,
Two among our number hold only nettle wine,
Three of us are killers, waiting hidden in line.
Choose, unless you wish to stay here forevermore,
To help you in your choice, we give you these clues four:
First, however slyly the poison tries to hide
You will always find some on nettle wine's left side;
Second, different are those who stand at either end,
But if you would move onward, neither is your friend;
Third, as you see clearly, all are different size,
Neither dwarf nor giant holds death in their insides;
Fourth, the second left and the second on the right
Are twins once you taste them, though different at first sight.

We can try to solve the riddle, though the lack of figures accompanying the problem means some difficulties. The story then tells that the children solved the problem and they found the liquid in the bottle on the right end that helped them go through the door back. The liquid which helped them go through the door leading to the following room, they found in the smallest bottle.

TASK: LET US PREPARE THE MISSING FIGURE OF THE BOOK!

Many questions can accompany the task. Through answering these questions we can draw a possible figure.

1. question: If we choose a bottle randomly and drink its content, what is the probability of choosing the one that helps us through the door leading to the following room?
1. answer: As there is only one such bottle (of 7), the probability of choosing the right one is 1/7.

2. question: If we chose a bottle randomly and drink its content, what is the probability of avoiding a sudden death?
2. answer: As two bottles contain poison out of seven, that is why 5 bottles mean no harm, so the probability of a chosen bottle not containing poison is 5/7.

As these chances augur ill, it is worth going on with thinking.

3. question: How many variations are there for lining up 7 different bottles?
3. answer: As we can choose a bottle for the first position in seven ways, for the second position in 6 ways, for the third position in five ways, for the fourth position in 4 ways, for the fifth position in 3 ways, for the sixth position in 2 ways and for the seventh position in one way, the number of possibilities is: $7{\cdot}6{\cdot}5{\cdot}4{\cdot}3{\cdot}2{\cdot}1= 5040$.

4. question: It is obvious that the content of the bottles is more important to us. How many variations are there for lining up the bottles if we consider only their content, that is, the 3 poisons and the 2 wines are considered identical?
4. answer: Within all the possible orders of the 7 bottles the first wine stands in front in half of the cases, and the other wine stands in front in the other half. So the sequence of $7{\cdot}6{\cdot}5{\cdot}4{\cdot}3{\cdot}2{\cdot}1$ has to be divided by 2, if we do not consider the order of the two wines. Similarly, the 3 poisons can have $3{\cdot}2{\cdot}1 = 6$ different orders, so the 3 poisons stand in a different position in the line of seven in one-sixth of the cases. The other cases would differ in the order of the 3 poisons, but we do not consider them now. So, considering the content of the bottles, the number of possible orders is: $\dfrac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot3\cdot2\cdot1} = 420$.

As a next step, we reduce the number of possible cases by considering more and more advice included in the riddle.

Let us start with the first one: „ First, however slyly the poison tries to hide You will always find some on nettle wine's left side" First we have to agree what left side means, which depends on the angle we look the bottles at. Let us fix, that the bottles are lined up on a table by the wall, this way, there is only one way to look at them in the front, and there is always a poison left to the wines. (The contrary case can also be studied.)

The following notation will be used for the sake of shortness:

P – poison, A – liquid that helps to go ahead, B – liquid that helps to go back, W – wine.

5. question: How many possible orders of the bottles are there, if there is always a poison on the left to a wine?
5. answer: As there is always a poison left to the wine, we consider each two bottles standing next to each other, containing poison and wine from left to right, respectively, as one. The bottles to be ordered this way are: PW, PW, P, A, B. These are five elements, there are two equal ones, so, similar to the above case, the number of possible orders is: $\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2}=60$.
6. question: If the bottles on the two sides of the line of bottles contain different kinds of liquids, how many possible orders of the bottles' contents are there?
6. answer: The content of the two side bottles is equal only if both are poisons, so PW is at the first position, and P is at the last one. The elements PW, A, B can be placed to the middle places in $3\cdot 2\cdot 1 = 6$ ways, so the number of orders where the two side elements are equal is: 6. If we subtract it from the previous 60 possibilities, we get the result that there are 54 orders where the content of the two side bottles are different.
7. question: If none of the two side bottles can be the A, then how many possibilities remained from the above 54?
7. answer: Let us consider the following cases depending on both the left and the right hand side of the line of the bottles:

If there is PW at the first and at the last position, then the 3 different elements in the middle positions can be ordered in 6 different ways. PW (A, B, P) PW

If the two side ones are B and PW, then they have two different orders, the 3 different elements in the middle have 6 orders, this makes $6\cdot 2=12$ cases. PW (PW, A, P) B

If the two side ones are the B and P, then these have two different orders, 2 out of the 3 elements in the middle are the same, that is why they have 3 orders, so this makes $3\cdot 2=6$ cases. B (PW, PW, A) P

If the two side ones are the P and PW, then they can have only one order, as both side bottles can not contain poison. The 3 elements in the middle can have 6 different orders that result in 6 cases.

P (PW, A, B) PW

There are altogether 30 ways of ordering the bottles.

The third advice does not mean any help yet. Let us consider the fourth one!

8. question: If the two second ones are the same if we look at them from the right and from the left hand side, then how many possibilities from the above 30 remain?
8. answer: Let us consider the cases according to the content of the bottle at the second position from the left and from the right side. If this is a wine, then neither A, nor an other P can stand there, only the following possibility remains:
   P W (A, B) P W B those in the brackets are permutable, that is why it means two cases.
   Poison can stand here if both are PW, in this case a P or a B can be at the start, and the two bottles can take up two different orders between the two PW's, this makes $2 \cdot 2=4$ cases.
   It is possible, that the second one from the left is a P, the second one from the right is a PW, this means, that only a B can stand on the left side, because it cannot be an A, and there is not enough space for a PW. In this case A and PW can have two different orders in the middle.

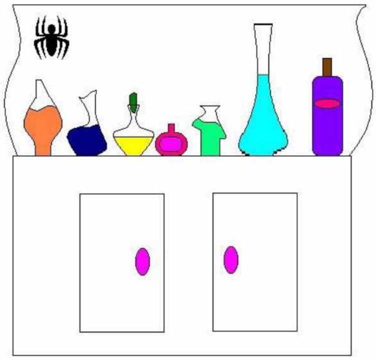
The contrary of the previous case is when the second one from the left is a PW, the second one from the right is a P, then there are two places left in the middle and one-one places at the sides, so the other PW is possible to be put only to the middle, that means A goes only to the sides, which is not possible, so this case can not occur.

Hence, by considering all advice, we obtained 8 cases concerning the content of the bottles. These are the following:

P W A P P W B
P W P A P W B
P P W A B P W
P P W B A P W
B P W A P P W
B P W P A P W
B P A P W P W
B P P W A P W

9. question: Applying the information that the smallest and the larges bottle cannot contain poison, let's draw a picture that makes possible to find out which bottle contains the liquid that helps to go ahead and the liquid that helps to go back.
9. answer: Let us consider the first two lines of the previous solution: If we put the largest bottle to the second position from the right or from the left, it turns out that it can contain only wine. It means, the right hand side is definitely a B,

and A is in the smallest bottle. Consequently, the following figure is correct and provides the same solution as is written in the book, i.e. bottle on the right hand side contains the liquid helping to go back, and the smallest one contains the liquid helping to go ahead. (the smallest and the largest bottles are permutable, but in that case we get a solution different from the one given in the book).



By examining the 6 further possibilities we can see, regardless of how we use the smallest and the largest bottles to denote something that is not a poison, we can find those among the listed cases, in which these positions contain an A and a B, and also one where a W is standing at one of the positions. For this, it is enough to examine positions 1 and 3, 1 and 4, 1 and 5, 3 and 4, 3 and 5, and 4 and 5.

Notes:

1. There is no other solution even if drinking were allowed of two bottles.
2. The English riddle is solvable without the figure, because of the meaning of the word „twins", so the second one from the left and from the right cannot be a poison (as there are three poisons), only a wine.
3. If we interpret "left side" reversely, then our solutions have to be adapted as well.

Methodological remarks:

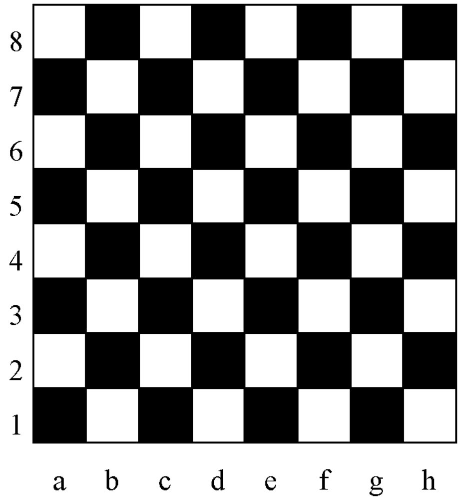
1. The above problems develop the combinatorial thinking of students, as several calculation methods are present in them: separation of cases, permutation, and repeated permutation.
2. It shows an example of how mathematical tasks are construed for topics interesting for children.
3. It directs attention to the fact, that in case of translating something to another language, it is more important to grasp the meaning of ideas than to make word-by-word translations.

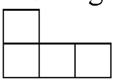
11. Chess board (Tamás Tárki)

In this sheet, we collected problems that, to some extent, are in connection with the chess board. They improve combinatorial thinking as well as they require special ideas, hence they develop creative thinking. Problems presented here do not require deep mathematical knowledge, so they are suitable for children to acquire mathematical thinking in a playful way. One of the methodological advantages of chessboard tasks is the possibility for making generalisations. The problems are generalisable in various directions, and it makes them suitable for being taught at every level of education. We find important that the problems are to be solved on the chessboard. This is a playful way for the children to learn and to discover relations. To reach this aim, it is needed to leave enough time for solving the given problems! It is not necessary to know the rules of chess to be able to solve the problems. What children should know is only the way how the different chess figures can move. The beginning of the task sheet contains simple questions. By answering these questions, we can teach pupils for orientation in the coordinate system, since the chessboard can be considered as a special coordinate system. These problems are followed by other problems improving divergent thinking. Some of them are simple counting tasks; others refer to finding a suitable construction. We presented colouring problems in the third part. The solution of these problems makes use of the colouring of the chessboard. We also present a problem where a different kind of colouring provides the required result. The solution of colouring problems might seem to be difficult, that is why it is useful to guide children towards the solution with the help of teachers' questions and hints.

11.1. The chessboard

The chessboard is an 8 × 8 size board, the squares of which are coloured as it is presented on the figure. The rows are marked from 1 to 8, the columns are marked by letters **a** to **h**. The square or square a1 is the lower left corner of the board and it is always black.



1. The knight can move from a given square to any direction in an L-shape way: . Define the squares the knight can step on, if it is standing on square c3!

Solution

The knight can move from c3 to the following squares: a2, a4, b5, d5, e4, e2, d1, b1. We obtain the squares if we increase or decrease the number of rows by 1 and the number of columns by 2, or vice versa, change the number of rows by 2 and the number of columns by 1.

2. The bishop can move only diagonally, though it can make any number of steps within. Define the squares the bishop can step on if it is standing on square e2!

Solution

The bishop can move from e2 to the following squares: d1, f3, g4, h5, a6, b5, c4, d3, f1.

3. The castle can move in row or column, and can skip any number of squares. Which squares can the castle step on, if it stays on square d5?

Solution

The castle can move from square d5 to the following squares: a5, b5, c5, e5, f5, g5, h5, d1, d2, d3, d4, d6, d7, d8.

4. You can move according to the regulations of the queen, the castle or the bishop. Which squares can the queen move to if it is standing on square c3?

Solution
The queen can move from c3 to the following squares: c1, c2, c4, c5, c6, c7, c8, a3, b3, d3, e3, f3, g3, h3, a1, b2, d4, e5, f6, g7, h8, a5, b4, d2, e1.

11.2. Combinatorial problems

1. Find the square from which the knight can step to as few squares as possible!

Solution
It is evident, that the required square should be at the corner of the board. There are four such squares: a1, a8, h1, h8. The knight can move only to 2 squares from these squares. (For example, from a1 it can move only to b3 or c2.) After solving the problem, ask pupils to find the square from which the knight can move to the most possible squares! It could be useful to fill in the 64 squares of a chessboard in such way, that we write the number of other squares the knight can step from there into them.

2. Find the square from which the castle can move to as few squares as possible!

Solution
Whichever square the castle is standing on, it can step on 7 squares in its row and in its column as well. So it can reach 14 other squares from each square.

3. Which is the square from which the queen can step on the least number of squares? Which is the square from which it can reach the most number of squares?

Solution
The queen can move to the least number of squares from the side squares of the board. These squares are the bordering squares of an 8×8 size square. From a square like this it can reach 21 squares altogether. If we consider squares next to these, we get the bordering squares of a 7×7 size square. With the queen, 23 squares can be reached from each of them. If we move according to this towards the centre of the table, the number of reachable squares will increase with 2 each time. On the following table, we wrote the number of places the queen can move to from the given square. The most number of squares can be reached from the four central squares (centre) of the chessboard, exactly 27.

The number of possible moves by the queen

21	21	21	21	21	21	21	21
21	23	23	23	23	23	23	21
21	23	25	25	25	25	23	21
21	23	25	27	27	25	23	21
21	23	25	27	27	25	23	21
21	23	25	25	25	25	23	21
21	23	23	23	23	23	23	21
21	21	21	21	21	21	21	21

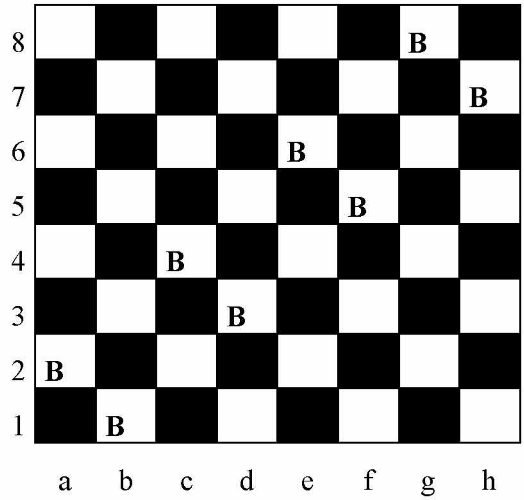
4. At most how many castles can be placed on the chessboard without any two of them taking each other?

Solution

It is self-evident, that there can not be more than 8 castles placed; otherwise there would be at least two castles in one of the rows mutually taking each other. It is easy to see, that 8 castles can be placed, because if we place them into one of the main diagonals of the board, they can not take each other. The problem has several other solutions. We present two of them in the following figures. Be sure to look for further solutions with the children!

Castles that cannot take each other

8							**B**	
7						**B**		
6					**B**			
5				**B**				
4			**B**					
3		**B**						
2	**B**							
1								**B**
	a	b	c	d	e	f	g	h

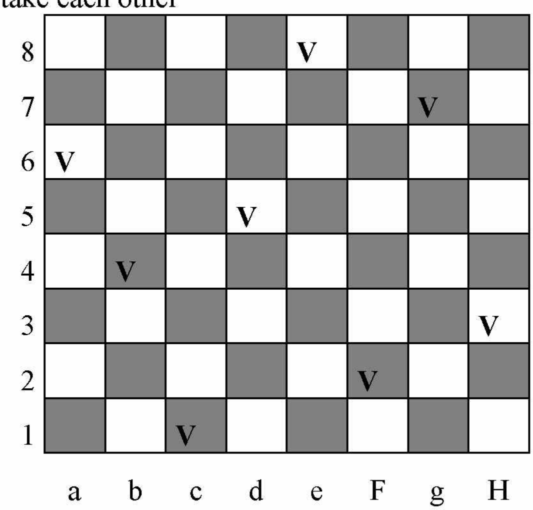


5. We get a much more difficult version of the previous problem, if we set it with the queen; how many queens can be placed on the chessboard without any two of them taking each other?

Solution

It is easy to see, that there cannot be more than 8 queens. It is not easy to show that 8 queens can be placed on the board. Note: 8 queens can be placed on the board without taking each other in 92 ways. We present one possible way here.

Queens that do not take each other



6. Put a king on square d8. The king can move to any neighbouring squares. It can reach the first row in 7 moves, if it goes only "downwards" in each move. Of course, if it moves "backwards" also, this way can be much longer. That is, to reach the first row, the 7 long way is the shortest. Define the number of shortest ways, if the king moves from square d8 to b1!

Solution

Because of its difficult nature, we suggest to use this problem only with groups of good ability, in spite of the fact that the solution we present does not require deep mathematical knowledge. One of the most interesting methodological lessons of the problem: if we measure distance on the chessboard by the number of steps, then the shortest way does not have uniqueness, unlike in Euclidean Geometry.

Consider the following table, where in each square, we have included the number of ways the king can reach that square, if it moves only "downwards", that is towards the first row. It can reach only 3 squares in the 7^{th} row, and it can reach each of the three in one way. It can reach 5 squares already in row 6. The two sides can be reached in one – one way. The other squares of row 6 are to be filled in as follows: For example, the king can arrive to square d6 from c7 (where it could go only in one way), it can arrive from d7 (where it could arrive in one way), and can arrive from e7 (where it could arrive only in one way). It means 1+1+1=3 ways to go to d7. Then, it is easy to formulate the rules of filling the board in: each square contains the sum of numbers in squares one row above the given row, touching the square along vertex or edge. According to the table, the king can reach square b1 in 259 ways, supposing that it takes the shortest, that is 7 steps long way.

The number of shortest ways the king can move on

8				K				
7			1	1	1			
6		1	2	3	2	1		
5	1	3	6	7	6	3	1	
4	4	10	16	19	16	10	4	1
3	14	30	45	51	45	30	15	5
2	44	89	126	141	126	90	50	20
1	133	259	356	393	357	266	160	70
	a	b	c	d	e	f	g	h

11.3. Colouring problems

1. Can the knight reach square h8 from square a1, if it touches each square exactly once?

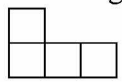
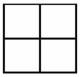
Solution

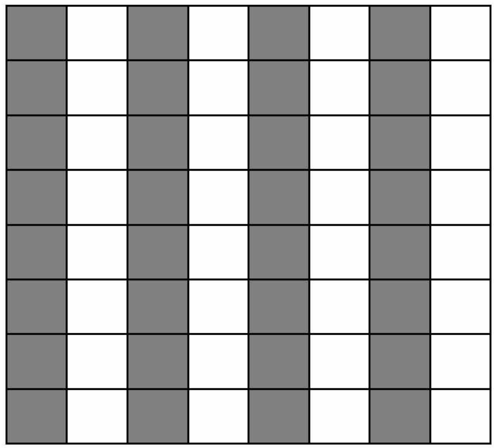
In each move, the knight goes to a square of opposite colour than the square it started from. To reach square h8 while it touches each square exactly once takes 63 moves. This is an odd number of many moves, so at the end of the way, it reaches a square of opposite colour than it started from. As a1 and h8 are both dark coloured, this "journey" of the knight is impossible.

2. Cover the chessboard with 1×2 size dominoes! Is it possible to cover the board if we leave squares a1 and h8 out?

Solution

It is easy to cover the 8×8 size board, if we go on with the "flooring" row by row. We can observe that each domino covers a black and a white square. That is why in each case when we cover the board with 1×2 size dominoes, the number of black and white squares covered is the same. If we leave the two side squares of the main diagonal out, we leave squares of the same colour out, so the number of black and white squares will be different on this incomplete board, and this way the covering is impossible to be done.

3. Somebody has repainted the chessboard as it is depicted in the figure below. Is it possible to cover this repainted board with 15 pieces of  shape, and 1 piece of  shape tetraminoes?



Solution

From methodological aspects, we suggest to set this problem only after solving the previous two ones. We find important that the children should formulate guesses only after the actual trial. The previous problems have taught us a lesson that we should think at the problem, that the number of black and white squares covered can not be equal. The one square-shaped tetramino covers 2 black and 2 white squares. We should cover the remaining 30-30 squares with L-shaped tetraminos. We can see that an L-shape like this can cover 1 or 3 black squares, depending on its position. As there are 15 pieces at our disposal, there are even many pieces from one type, and odd many pieces from the other, as an odd number can be divided to the sum of two integers only this way. Suppose we have even number of pieces from the type covering 1 grey and 3 whites, and odd number of pieces from the type covering 3 greys and 1 white. Then the number of black squares covered is:

$1 \cdot \text{even} + 3 \cdot \text{odd} = \text{odd}$, which is not possible, as 30 is an even number.

The number of black squares covered in the contrary case: $1 \cdot \text{odd} + 3 \cdot \text{even} = \text{odd}$, that is also not possible. This way we obtained, that the covering is not possible.

References

ENGEL, A. (1998): Problem-solving strategies. Spinger

12. Sport games (Tamás Árki)

12.1. Problem

Figure 1 shows the sketch of the most beautiful attack in the school football championship. Four forwards participated in the attack: **A**lbert, **B**ill, **C**edric and **D**oug. We have marked the ball's course on the figure. Who kicked the goal? Who gave the goal pass? Who launched the attack?

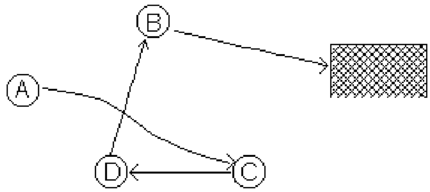


Figure 1

Solution

We can follow the course of the attack along the arrows. The ball started from Albert, then by touching Cedric it reached Doug, who passed it to Bill, and he realised Doug's goal pass. In Figure 1, points A, B, C and D and the directed lines consist a (directed) graph together. The points are called the vertices of the graph, and the lines are called the edges of the graph. Consider, for example, some graphs from everyday life! We can think of the road system of a country (here the crossings and towns are the vertices, and the roads themselves are the edges). Or, we can imagine a telephone network, where the telephone exchanges and subscribers are the vertices, and lines are the edges.

12.2. Problem

What is the number of all the ways, such that the ball can go to the gate in the previous problem, if the attack starts from Albert and each of the four forwards can touch it exactly once?

Solution

Albert can pass the ball to any of his three companions, then the player who got the ball can choose from two companions, finally the last but one player does not have a possibility to choose. We can sum the given cases up in the graph in Figure 2, where we can follow the course of the ball in the given attacks by going along the sections. The total number of cases is $3{\cdot}2{=}6$. Prepare the graph model belonging to the given cases, as learnt in the previous problem! Solve the task also in the case, when we do not define that the attack should start from Albert!

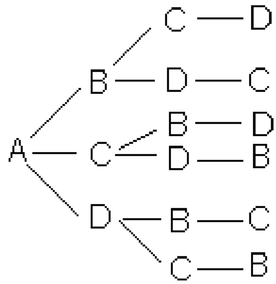


Figure 2

12.3. Problem

We would like to line up all the 11 members of a football team. How many ways can it be done altogether?

Solution

Based on the ideas of the previous problem, we can again prepare the graph of the line-ups. Observe, that we can put any of the 11 players to the first position, we can put any of the remaining 10 players to the following position, and so on; and we can put only 1 player to the last position. The total number of all orders are: $11\cdot 12\cdots 2\cdot 1=39916800$. The product on the left side is called 11 factorial, and shortly write as $11!$.

12.4. Problem

The result became a draw on a football game, so the game should be settled with the help of penalty kicks. Five players kick a penalty in the first round. In how many orders can one of the teams kick the penalties, if any of the 10 midfield players can make the kick, and the order of the kickers is also important?

Solution

Any of the 10 players can kick the first penalty, the second one can be kicked only by 9 players, and so on; the fifth penalty can be kicked by any of the remaining 6 players. The total number of possibilities is $10\cdot 9\cdot 8\cdot 7\cdot 6=30240$.

12.5. Problem

Nine teams took part in a school basketball championship. The championship was done in an elimination system; the drawn teams played a game and the winner of this game moved on. In case the score was even, they decided with the help of penalty throws who should move on. If a team didn't get an opponent in the drawing process, that team moved on without playing a game, as a "power-winner". How many games did the teams play altogether?

Solution

A possible championship flow is depicted on Figure 3. Black circles mark the teams; the matches are marked by letters "V". The last team had an opponent only in the last round, and reached that round as a power-winner. The number of matches, as the figure shows, is 8.

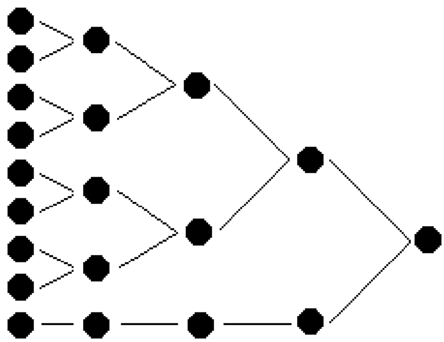


Figure 3

12.6. Problem

In the arrangement system of the previous problem, how many matches are played by the teams, if n teams entered the championship?

Solution

If we observe from Figure 3, that one team is eliminated from the championship during each match. That is why the number of matches played is as much as the number of eliminated teams is. As only the winning team is left by the end, the number of matches is $n-1$.

12.7. Problem

In the arrangement system of the previous problem, a round means the number of matches played at the same time. How many matches were played in the championship, if there could be more power-winners in the given rounds?

Observe that our previous track of thought applies here as well, hence it is still true, that one team is eliminated during each match, and a team can be eliminated only after a finished game. The number we look for is again $n-1$.

12.8. Problem

In the arrangement system of problem 5, there cannot be more than one power-winner team in a round. If there are n teams taking part in a round, then how many matches consist of a round?

Solution

If n is even, then each team has an opponent, so the number of matches is $\frac{n}{2}$. If n is odd, then there is one power-winner team, and each of the remaining $n-1$ teams has an opponent, so the number of matches is $\frac{n-1}{2}$. This number equals to $\left[\frac{n}{2}\right]$ in both cases. ([x] is the biggest integer that is not bigger than x.)

12.9. Problem

Under the arrangements of the previous problem, if there are n teams playing in a round, then how many teams enter the next round?

Solution

If n is even, then each team has an opponent, so the number of teams eliminated, and the number of teams entering the next round is $\frac{n}{2}$. If n is odd, then one team enters the next round as a power-winner, half of the remaining $n-1$ teams is eliminated, and half of them moves on, so the number of teams entering the next round is $1+\frac{n-1}{2}=\frac{n+1}{2}$. This number equals with $\left[\frac{n+1}{2}\right]$ in both cases.

12.10. Problem

Count the sum below as "skilfully" as possible, that is, apply as little calculation as possible! Generalise the problem!

$$\left[\frac{17}{2}\right]+\left[\frac{18}{4}\right]+\left[\frac{20}{8}\right]+\left[\frac{24}{16}\right]+\left[\frac{32}{32}\right].$$

Solution

Examine a one-round, elimination championship of 17 teams!

First round

Number of matches: $\left[\frac{17}{2}\right]=8$;

Number of those qualified to enter the next round:

$\left[\frac{17+1}{2}\right]=\left[\frac{18}{2}\right]=9$.

Second round

Number of matches: $\left[\frac{\left[\frac{18}{2}\right]}{2}\right]=\left[\frac{18}{4}\right]=4$;

Number of those qualified to enter the next round:

$\left[\frac{\left[\frac{18}{2}\right]+1}{2}\right]=\left[\frac{20}{4}\right]=5$.

Third round

Number of matches: $\left[\frac{\left[\frac{20}{4}\right]}{2}\right]=\left[\frac{20}{8}\right]=2$;

Number of those qualified to enter the next round:

$\left[\frac{\left[\frac{20}{4}\right]+1}{2}\right]=\left[\frac{24}{8}\right]=3$.

Fourth round

Number of matches: $\left[\frac{\left[\frac{24}{8}\right]}{2}\right]=\left[\frac{24}{16}\right]=1$;

Number of those qualified to enter the next round:

$\left[\frac{\left[\frac{24}{8}\right]+1}{2}\right]=\left[\frac{32}{16}\right]=2$.

Fifth round

Number of matches: $\left[\frac{\left[\frac{32}{16}\right]}{2}\right]=\left[\frac{32}{32}\right]=1$;

Number of those qualified to enter the next round:

$\left[\frac{\left[\frac{32}{16}\right]+1}{2}\right]=\left[\frac{48}{32}\right]=1$.

It is visible, that the sum we are looking for is exactly the number of matches played in a one-round, elimination championship of 17 teams, that is 16.
The problem is generalisable. Its generalisation is:

$$\left[\frac{n}{2}\right]+\left[\frac{n+1}{4}\right]+\left[\frac{n+3}{8}\right]+\ldots+\left[\frac{n+2^{k-1}-1}{2^k}\right]+\ldots=n-1.$$

Of course only a finite sum can be on the left side, hence counting from somewhere (suitably starting from big *k*), each member is 0.

12.11. Problem

5 teams entered a basketball championship, and the teams played an all-against-all tournament, with return matches, that is each team played twice with the other. Define the total number of matches played by the teams! Solve the task if *n* teams entered the championship!

Solution

We can summarise the results of the championship in a table. Let's mark the teams with letter A, B, C, D, and E, for the sake of simplicity. We wrote the results obtained by a match between teams *x* and *y* into the cells (*x*, *y*).

	A	B	C	D	E
A	—	75 : 56	88 : 85	68 : 98	56 : 60
B	67 : 70	—	57 : 65	90 : 90	82 : 86
C	64 : 65	65: 65	—	68 : 45	76 : 76
D	72 : 76	90 : 80	80 : 67	—	67 : 70
E	60 : 45	68 : 67	80 : 80	90 : 96	—

We can see that the results of 4 matches are presented in each row of the table, and the number of rows is 5. So the total number of matches is 5·4=20.

If *n* teams take part in the championship, then in each *n* row of the table there are $n-1$ matches, so the number of matches is $n(n-1)$.

The task can be approached in the following way, as well. If we do not count the return matches, then the first team has played $n-1$ matches, the second team played also $n-1$ matches, but one match was already counted from this, so the number of "new", not yet counted matches is $n-2$. Similarly, the third team has played $n-3$ matches, and so on; the last but one team has played only 1 not yet counted match. So the total number of matches (not counting the return matches) is:

$$(n-1)+(n-2)+\ldots+2+1.$$

By comparing it to our previous result we get, that

$$1+2+\ldots+n-1=\frac{n(n-1)}{2},$$

so, by comparing our methods, we obtain a closed formula for the sum of the first $n-1$ positive integers.

12.12. Problem

In the all-against-all championship the winner of each match gets 2 points, the loser team does not get a point, in case of an equal score both teams get 1–1 point. Can the following number sequence mean the scores obtained by the teams in one round?

a. 0, 1, 1, 2, 2, 0, 1, 0, 2, 1, 2;
b. 0, 1, 1, 2, 2, 0, 0, 2, 0, 2, 1;

Solution

As there are 4 winner teams in case of a), and there are only 3 looser ones, this can not be the sequence of the obtained scores.
The solution of problem b) is the same; there can not be 3 teams who reached the same score.

12.13. Problem

We know the following statements about the teams of an all-against-all championship of 20 teams;

a) There is a team that won in an away game;
b) Choosing any two teams we find one, that has lost each of its away games;

Can we tell which team has won the championship?

Solution

Suppose that A is the team that, according to a), has won at least once an away game. Choose an optional B team different from A! Then, according to b), from A and B one has lost all of its away games, but this team can only be the B. So apart from team A, every other team has lost its away games. This also means, that each team has won its home games, apart possibly from the one played with A. So the championship could only be won by team A. Observe, that we did not use the number of teams among the prerequisites of the problem!

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